Appendix B

Linear stability

> Mopping up operations are the activities that engage most scientists throughout their careers.
>
> — Thomas Kuhn, *The Structure of Scientific Revolutions*

The subject of linear algebra generates innumerable tomes of its own, and is way beyond what we can exhaustively cover. Here we recapitulate a few essential concepts that ChaosBook relies on. The punch line (B.22):

Hamilton-Cayley equation $\prod(\mathbf{M}-\lambda_i \mathbf{1}) = 0$ associates with each distinct root λ_i of a matrix $\mathbf{M}$ a projection onto ith vector subspace

$$\mathbf{P}_i = \prod_{j \neq i} \frac{\mathbf{M} - \lambda_j \mathbf{1}}{\lambda_i - \lambda_j} .$$

B.1 Linear algebra

The reader might prefer going straight to sect. B.2.

Vector space. A set V of elements $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ is called a *vector* (or *linear*) *space* over a field $\mathbb{F}$ if

(a) *vector addition* "+" is defined in V such that V is an abelian group under addition, with identity element $\mathbf{0}$;

(b) the set is *closed* with respect to *scalar multiplication* and vector addition

$$\begin{aligned} a(\mathbf{x}+\mathbf{y}) &= a\mathbf{x} + a\mathbf{y} , \qquad a, b \in \mathbb{F} , \quad \mathbf{x}, \mathbf{y} \in V \\ (a+b)\mathbf{x} &= a\mathbf{x} + b\mathbf{x} \\ a(b\mathbf{x}) &= (ab)\mathbf{x} \\ 1\,\mathbf{x} &= \mathbf{x} , \qquad 0\,\mathbf{x} = \mathbf{0} . \end{aligned} \tag{B.1}$$

Here the field $\mathbb{F}$ is either $\mathbb{R}$, the field of reals numbers, or $\mathbb{C}$, the field of complex numbers. Given a subset $V_0 \subset V$, the set of all linear combinations of elements of V_0, or the *span* of V_0, is also a vector space.

A basis. $\{\mathbf{e}^{(1)}, \cdots, \mathbf{e}^{(d)}\}$ is any linearly independent subset of V whose span is V. The number of basis elements d is the *dimension* of the vector space V.

Dual space, dual basis. Under a general linear transformation $g \in GL(n, \mathbb{F})$, the row of basis vectors transforms by right multiplication as $\mathbf{e}^{(j)} = \sum_k (\mathbf{g}^{-1})^j{}_k \, \mathbf{e}^{(k)}$, and the column of x_a's transforms by left multiplication as $x' = \mathbf{g}x$. Under left multiplication the column (row transposed) of basis vectors $\mathbf{e}_{(k)}$ transforms as $\mathbf{e}_{(j)} = (\mathbf{g}^\dagger)_j{}^k \mathbf{e}_{(k)}$, where the *dual rep* $\mathbf{g}^\dagger = (\mathbf{g}^{-1})^T$ is the transpose of the inverse of $\mathbf{g}$. This observation motivates introduction of a *dual* representation space $\bar{V}$, the space on which $GL(n, \mathbb{F})$ acts via the dual rep $\mathbf{g}^\dagger$.

Definition. If V is a vector representation space, then the *dual space* $\bar{V}$ is the set of all linear forms on V over the field $\mathbb{F}$.

If $\{\mathbf{e}^{(1)}, \cdots, \mathbf{e}^{(d)}\}$ is a basis of V, then $\bar{V}$ is spanned by the *dual basis* $\{\mathbf{e}_{(1)}, \cdots, \mathbf{e}_{(d)}\}$, the set of d linear forms $\mathbf{e}_{(k)}$ such that

$$\mathbf{e}_{(j)} \cdot \mathbf{e}^{(k)} = \delta_j^k ,$$

where δ_j^k is the Kronecker symbol, $\delta_j^k = 1$ if $j = k$, and zero otherwise. The components of dual representation space vectors $\bar{y} \in \bar{V}$ will here be distinguished by upper indices

$$(y^1, y^2, \ldots, y^n) . \tag{B.2}$$

They transform under $GL(n, \mathbb{F})$ as

$$y'^a = (\mathbf{g}^\dagger)^a{}_b y^b . \tag{B.3}$$

For $GL(n, \mathbb{F})$ no complex conjugation is implied by the † notation; that interpretation applies only to unitary subgroups $U(n) \subset GL(n, \mathbb{C})$. $\mathbf{g}$ can be distinguished from $\mathbf{g}^\dagger$ by meticulously keeping track of the relative ordering of the indices,

$$(\mathbf{g})_a^b \rightarrow g_a{}^b , \qquad (\mathbf{g}^\dagger)_a^b \rightarrow g^b{}_a . \tag{B.4}$$

Algebra. A set of r elements $\mathbf{t}_\alpha$ of a vector space $\mathcal{T}$ forms an algebra if, in addition to the vector addition and scalar multiplication,

(a) the set is *closed* with respect to multiplication $\mathcal{T} \cdot \mathcal{T} \to \mathcal{T}$, so that for any two elements $\mathbf{t}_\alpha, \mathbf{t}_\beta \in \mathcal{T}$, the product $\mathbf{t}_\alpha \cdot \mathbf{t}_\beta$ also belongs to $\mathcal{T}$:

$$\mathbf{t}_\alpha \cdot \mathbf{t}_\beta = \sum_{\gamma=0}^{r-1} \tau_{\alpha\beta}{}^\gamma \mathbf{t}_\gamma \,, \qquad \tau_{\alpha\beta}{}^\gamma \in \mathbb{C} \,; \tag{B.5}$$

(b) the multiplication operation is *distributive*:

$$\begin{aligned} (\mathbf{t}_\alpha + \mathbf{t}_\beta) \cdot \mathbf{t}_\gamma &= \mathbf{t}_\alpha \cdot \mathbf{t}_\gamma + \mathbf{t}_\beta \cdot \mathbf{t}_\gamma \\ \mathbf{t}_\alpha \cdot (\mathbf{t}_\beta + \mathbf{t}_\gamma) &= \mathbf{t}_\alpha \cdot \mathbf{t}_\beta + \mathbf{t}_\alpha \cdot \mathbf{t}_\gamma \,. \end{aligned}$$

The set of numbers $\tau_{\alpha\beta}{}^\gamma$ are called the *structure constants*. They form a matrix rep of the algebra,

$$(\mathbf{t}_\alpha)_\beta{}^\gamma \equiv \tau_{\alpha\beta}{}^\gamma \,, \tag{B.6}$$

whose dimension is the dimension of the algebra itself.

Depending on what further assumptions one makes on the multiplication, one obtains different types of algebras. For example, if the multiplication is associative

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta) \cdot \mathbf{t}_\gamma = \mathbf{t}_\alpha \cdot (\mathbf{t}_\beta \cdot \mathbf{t}_\gamma) \,,$$

the algebra is *associative*. Typical examples of products are the *matrix product*

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta)_a^c = (t_\alpha)_a^b (t_\beta)_b^c \,, \qquad \mathbf{t}_\alpha \in V \otimes \bar{V} \,, \tag{B.7}$$

and the *Lie product*

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta)_a^c = (t_\alpha)_a^b (t_\beta)_b^c - (t_\alpha)_c^b (t_\beta)_b^a \,, \qquad \mathbf{t}_\alpha \in V \otimes \bar{V} \tag{B.8}$$

which defines a *Lie algebra*.

B.2 Eigenvalues and eigenvectors

Eigenvalues of a $[d \times d]$ matrix $\mathbf{M}$ are the roots of its characteristic polynomial

$$\det(\mathbf{M} - \lambda \mathbf{1}) = \prod (\lambda_i - \lambda) = 0 \,. \tag{B.9}$$

Given a nonsingular matrix $\mathbf{M}$, with all $\lambda_i \neq 0$, acting on d-dimensional vectors $\mathbf{x}$, we would like to determine *eigenvectors* $\mathbf{e}^{(i)}$ of $\mathbf{M}$ on which $\mathbf{M}$ acts by scalar multiplication by eigenvalue λ_i

$$\mathbf{M}\mathbf{e}^{(i)} = \lambda_i \mathbf{e}^{(i)} \,. \tag{B.10}$$

If $\lambda_i \neq \lambda_j$, $\mathbf{e}^{(i)}$ and $\mathbf{e}^{(j)}$ are linearly independent, so there are at most d distinct eigenvalues, which we assume have been computed by some method, and ordered by their real parts, $\mathrm{Re}\,\lambda_\mathrm{i} \geq \mathrm{Re}\,\lambda_{\mathrm{i+1}}$.

If all eigenvalues are distinct $\mathbf{e}^{(j)}$ are d linearly independent vectors which can be used as a (non-orthogonal) basis for any d-dimensional vector $\mathbf{x} \in \mathbb{R}^d$

$$\mathbf{x} = x_1 \, \mathbf{e}^{(1)} + x_2 \, \mathbf{e}^{(2)} + \cdots + x_d \, \mathbf{e}^{(d)} \,. \tag{B.11}$$

From (B.10) it follows that matrix $(\mathbf{M} - \lambda_i \mathbf{1})$ annihilates $\mathbf{e}^{(i)}$,

$$(\mathbf{M} - \lambda_i \mathbf{1})\mathbf{e}^{(j)} = (\lambda_j - \lambda_i)\mathbf{e}^{(j)} \,,$$

and the product of all such factors annihilates any vector, so the matrix $\mathbf{M}$ satisfies its characteristic equation (B.9),

$$\prod_{i=1}^{d} (\mathbf{M} - \lambda_i \mathbf{1}) = 0 \,. \tag{B.12}$$

This humble fact has a name: the Hamilton-Cayley theorem. If we delete one term from this product, we find that the remainder projects $\mathbf{x}$ onto the corresponding eigenvector:

$$\prod_{j \neq i} (\mathbf{M} - \lambda_j \mathbf{1})\mathbf{x} = \prod_{j \neq i} (\lambda_i - \lambda_j) x_i \mathbf{e}^{(i)} \,.$$

Dividing through by the $(\lambda_i - \lambda_j)$ factors yields the *projection operators*

$$\mathbf{P}_i = \prod_{j \neq i} \frac{\mathbf{M} - \lambda_j \mathbf{1}}{\lambda_i - \lambda_j} \,, \tag{B.13}$$

which are *orthogonal* and *complete*:

$$\mathbf{P}_i \mathbf{P}_j = \delta_{ij} \mathbf{P}_j \,, \quad (\text{no sum on } j) \,, \qquad \sum_{i=1}^{r} \mathbf{P}_i = \mathbf{1} \,. \tag{B.14}$$

By (B.10) every column of $\mathbf{P}_i$ is proportional to a right eigenvector $\mathbf{e}^{(i)}$, and its every row to a left eigenvector $\mathbf{e}_{(i)}$. In general, neither set is orthogonal, but by the idempotence condition (B.14), they are mutually orthogonal,

$$\mathbf{e}_{(i)} \cdot \mathbf{e}^{(j)} = c\,\delta_i^j \,. \tag{B.15}$$

The non-zero constant c is convention dependent and not worth fixing, unless you feel nostalgic about Clebsch-Gordan coefficients. It follows from the characteristic equation (B.12) that λ_i is the eigenvalue of $\mathbf{M}$ on $\mathbf{P}_i$ subspace:

$$\mathbf{M}\mathbf{P}_i = \lambda_i \mathbf{P}_i \qquad (\text{no sum on } i)\,. \tag{B.16}$$

Using $\mathbf{M} = \mathbf{M}\,\mathbf{1}$ and completeness relation (B.14) we can rewrite $\mathbf{M}$ as

$$\mathbf{M} = \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \cdots + \lambda_d \mathbf{P}_d \,. \tag{B.17}$$

Any matrix function $f(\mathbf{M})$ takes the scalar value $f(\lambda_i)$ on the $\mathbf{P}_i$ subspace, $f(\mathbf{M})\mathbf{P}_i = f(\lambda_i)\mathbf{P}_i$, and is easily evaluated through its *spectral decomposition*

$$f(\mathbf{M}) = \sum_i f(\lambda_i)\mathbf{P}_i \,. \tag{B.18}$$

This, of course, is the reason why anyone but a fool works with irreducible reps: they reduce matrix (AKA "operator") evaluations to manipulations with numbers.

Example B.1 Complex eigenvalues. *As $\mathbf{M}$ has only real entries, it will in general have either real eigenvalues, or complex conjugate pairs of eigenvalues. That is not surprising, but also the corresponding eigenvectors can be either real or complex. All coordinates used in defining the flow are real numbers, so what is the meaning of a* complex *eigenvector?*

If λ_k, λ_{k+1} eigenvalues that lie within a diagonal $[2\times 2]$ sub-block $\mathbf{M}' \subset \mathbf{M}$ form a complex conjugate pair, $\{\lambda_k, \lambda_{k+1}\} = \{\mu + i\omega, \mu - i\omega\}$, the corresponding complex eigenvectors can be replaced by their real and imaginary parts, $\{\mathbf{e}^{(k)}, \mathbf{e}^{(k+1)}\} \to \{\mathrm{Re}\,\mathbf{e}^{(k)}, \mathrm{Im}\,\mathbf{e}^{(k)}\}$. In this 2–$d$ real representation the block $\mathbf{M}' \to \mathbf{N}$ consists of the identity and the generator of $SO(2)$ rotations

$$\mathbf{N} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} = \mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Trajectories of $\dot{\mathbf{x}} = \mathbf{N}\,\mathbf{x}$, $\mathbf{x}(t) = J^t\,\mathbf{x}(0)$, where

$$J^t = e^{t\mathbf{N}} = e^{t\mu} \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} , \tag{B.19}$$

spiral in/out around $(x,y) = (0,0)$, see figure 4.4, with the rotation period T and the expansion/contraction multiplier along the $\mathbf{e}^{(j)}$ eigendirection per a turn of the spiral:

B.1
page 724

$$T = 2\pi/\omega\,, \qquad \Lambda_{radial} = e^{T\mu}\,, \qquad \Lambda_j = e^{T\mu^{(j)}} \,. \tag{B.20}$$

We learn that the typical turnover time scale in the neighborhood of the equilibrium $(x,y) = (0,0)$ is of order $\approx T$ (and not, let us say, $1000\,T$, or $10^{-2}T$). Λ_j multipliers give us estimates of strange-set thickness.

While for a randomly constructed matrix all eigenvalues are distinct with probability 1, that is not true in presence of symmetries. What can one say about situation where d_α eigenvalues are degenerate, $\lambda_\alpha = \lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+d_\alpha-1}$? Hamilton-Cayley (B.12) now takes form

$$\prod_{\alpha=1}^{r} (\mathbf{M} - \lambda_\alpha \mathbf{1})^{d_\alpha} = 0\,, \qquad \sum_\alpha d_\alpha = d\,. \tag{B.21}$$

We distinguish two cases:

M can be brought to diagonal form. The characteristic equation (B.21) can be replaced by the minimal polynomial,

$$\prod_{\alpha=1}^{r} (\mathbf{M} - \lambda_\alpha \mathbf{1}) = 0\,, \tag{B.22}$$

where the product includes each distinct eigenvalue only once. Matrix $\mathbf{M}$ satisfies

$$\mathbf{M}\mathbf{e}^{(\alpha,k)} = \lambda_i \mathbf{e}^{(\alpha,k)}\,, \tag{B.23}$$

on a d_α-dimensional subspace spanned by a linearly independent set of basis eigenvectors $\{\mathbf{e}^{(\alpha,1)}, \mathbf{e}^{(\alpha,2)}, \cdots, \mathbf{e}^{(\alpha,d_\alpha)}\}$. This is the easy case whose discussion we continue in appendix H.2.1. Luckily, if the degeneracy is due to a finite or compact symmetry group, relevant $\mathbf{M}$ matrices can always be brought to such Hermitian, diagonalizable form.

M can only be brought to upper-triangular, Jordan form. This is the messy case, so we only illustrate the key idea in example B.2.

Example B.2 Decomposition of 2–d vector spaces: *Enumeration of every possible kind of linear algebra eigenvalue / eigenvector combination is beyond what we can reasonably undertake here. However, enumerating solutions for the simplest case, a general $[2\times 2]$ non-singular matrix*

$$\mathbf{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} .$$

takes us a long way toward developing intuition about arbitrary finite-dimensional matrices. The eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\mathrm{tr}\,\mathbf{M} \pm \frac{1}{2}\sqrt{(\mathrm{tr}\,\mathbf{M})^2 - 4\det\mathbf{M}} \tag{B.24}$$

are the roots of the characteristic (secular) equation

$$\begin{aligned}\det(\mathbf{M} - \lambda\,\mathbf{1}) &= (\lambda_1 - \lambda)(\lambda_2 - \lambda)\\ &= \lambda^2 - \mathrm{tr}\,\mathbf{M}\,\lambda + \det\mathbf{M} \;=\; 0\,.\end{aligned}$$

Distinct eigenvalues case *has already been described in full generality. The left/right eigenvectors are the rows/columns of projection operators*

$$P_1 = \frac{\mathbf{M} - \lambda_2\mathbf{1}}{\lambda_1 - \lambda_2}\,, \qquad P_2 = \frac{\mathbf{M} - \lambda_1\mathbf{1}}{\lambda_2 - \lambda_1}\,, \qquad \lambda_1 \neq \lambda_2\,. \tag{B.25}$$

Degenerate eigenvalues. *If $\lambda_1 = \lambda_2 = \lambda$, we distinguish two cases: (a) $\mathbf{M}$ can be brought to diagonal form. This is the easy case whose discussion in any dimension we continue in appendix H.2.1. (b) $\mathbf{M}$ can be brought to* Jordan *form, with zeros everywhere except for the diagonal, and some 1's directly above it; for a $[2\times 2]$ matrix the Jordan form is*

$$\mathbf{M} = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}, \qquad \mathbf{e}^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}, \quad \mathbf{v}^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}.$$

$\mathbf{v}^{(2)}$ helps span the 2–d space, $(\mathbf{M}-\lambda)^2\mathbf{v}^{(2)} = 0$, but is not an eigenvector, as $\mathbf{M}\mathbf{v}^{(2)} = \lambda\mathbf{v}^{(2)} + \mathbf{e}^{(1)}$. For every such Jordan $[d_\alpha \times d_\alpha]$ block there is only one eigenvector per block. Noting that

$$\mathbf{M}^m = \begin{pmatrix}\lambda^m & m\lambda^{m-1}\\ 0 & \lambda^m\end{pmatrix},$$

we see that instead of acting multiplicatively on $\mathbb{R}^2$, fundamental matrix $J^t = \exp(t\mathbf{M})$

$$e^{t\mathbf{M}}\begin{pmatrix}u\\v\end{pmatrix} = e^{t\lambda}\begin{pmatrix}u + tv\\ v\end{pmatrix} \tag{B.26}$$

picks up a power-low correction. That spells trouble (logarithmic term $\ln t$ if we bring the extra term into the exponent).

Example B.3 Projection operator decomposition in 2–d: *Let's illustrate how the distinct eigenvalues case works with the $[2\times 2]$ matrix*

$$\mathbf{M} = \begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}.$$

Its eigenvalues $\{\lambda_1, \lambda_2\} = \{5, 1\}$ are the roots of (B.24):

$$\det(\mathbf{M} - \lambda\mathbf{1}) = \lambda^2 - 6\,\lambda + 5 = (5 - \lambda)(1 - \lambda) = 0\,.$$

That $\mathbf{M}$ satisfies its secular equation (Hamilton-Cayley theorem) can be verified by explicit calculation:

$$\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}^2 - 6\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix} + 5\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

Associated with each root λ_i is the projection operator *(B.25)*

$$P_1 = \frac{1}{4}(\mathbf{M} - \mathbf{1}) = \frac{1}{4}\begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix} \tag{B.27}$$

$$P_2 = \frac{1}{4}(\mathbf{M} - 5\cdot\mathbf{1}) = \frac{1}{4}\begin{pmatrix}1 & -1\\ -3 & 3\end{pmatrix}. \tag{B.28}$$

Matrices $\mathbf{P}_i$ are orthonormal and complete, The dimension of the ith subspace is given by $d_i = \mathrm{tr}\,\mathbf{P}_i$; in case at hand both subspaces are 1-dimensional. From the characteristic equation it follows that $\mathbf{P}_i$ satisfies the eigenvalue equation $\mathbf{M}\,\mathbf{P}_i = \lambda_i\mathbf{P}_i$. Two consequences are immediate. First, we can easily evaluate any function of $\mathbf{M}$ by spectral decomposition

$$\mathbf{M}^7 - 3\cdot\mathbf{1} = (5^7 - 3)\mathbf{P}_1 + (1 - 3)\mathbf{P}_2 = \begin{pmatrix}58591 & 19531\\ 58593 & 19529\end{pmatrix}.$$

Second, as $\mathbf{P}_i$ satisfies the eigenvalue equation, its every column is a right eigenvector, and every row a left eigenvector. Picking first row/column we get the eigenvectors:

$$\begin{aligned}\{\mathbf{e}^{(1)}, \mathbf{e}^{(2)}\} &= \{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-3\end{pmatrix}\}\\ \{\mathbf{e}_{(1)}, \mathbf{e}_{(2)}\} &= \{(\,3 \quad 1\,), (\,1 \quad -1\,)\},\end{aligned}$$

with overall scale arbitrary.The matrix is not hermitian , so $\{\mathbf{e}^{(j)}\}$ do not form an orthogonal basis. The left-right eigenvector dot products $\mathbf{e}_{(j)}\cdot\mathbf{e}^{(k)}$, however, are orthonormal (B.15) by inspection.

B.3 Stability of Hamiltonian flows

(M.J. Feigenbaum and P. Cvitanović)

The symplectic structure of Hamilton's equations buys us much more than the incompressibility, or the phase space volume conservation alluded to in sect. 6.1. We assume you are at home with Hamiltonian formalism. If you would like to see the Hamilton's equations derived, Hamilton's original line of reasoning is retraced in sect. 31.1.1. The evolution equations for any p, q dependent quantity $Q = Q(q, p)$ are given by (14.32).

sect. 31.1.1

In terms of the Poisson brackets, the time evolution equation for $Q = Q(q, p)$ is given by (14.34). We now recast the symplectic condition (**??**) in a form convenient for using the symplectic constraints on M. Writing $x(t) = x' = [p', q']$ and the fundamental matrix and its inverse

$$M = \begin{pmatrix}\frac{\partial q'}{\partial q} & \frac{\partial q'}{\partial p}\\ \frac{\partial p'}{\partial q} & \frac{\partial p'}{\partial p}\end{pmatrix}, \qquad M^{-1} = \begin{pmatrix}\frac{\partial q}{\partial q'} & \frac{\partial q}{\partial p'}\\ \frac{\partial p}{\partial q'} & \frac{\partial p}{\partial p'}\end{pmatrix}, \tag{B.29}$$

we can spell out the symplectic invariance condition (**??**):

$$
\begin{aligned}
\frac{\partial q'_k}{\partial q_i}\frac{\partial p'_k}{\partial q_j} - \frac{\partial p'_k}{\partial q_i}\frac{\partial q'_k}{\partial q_j} &= 0 \\
\frac{\partial q'_k}{\partial p_i}\frac{\partial p'_k}{\partial p_j} - \frac{\partial p'_k}{\partial p_i}\frac{\partial q'_k}{\partial p_j} &= 0 \\
\frac{\partial q'_k}{\partial q_i}\frac{\partial p'_k}{\partial p_j} - \frac{\partial p'_k}{\partial q_i}\frac{\partial q'_k}{\partial p_j} &= \delta_{ij}\,.
\end{aligned}
\qquad \text{(B.30)}
$$

From (6.18) we obtain

$$
\frac{\partial q_i}{\partial q'_j} = \frac{\partial p'_j}{\partial p_i}\,, \quad \frac{\partial p_i}{\partial p'_j} = \frac{\partial q'_j}{\partial q_i}\,, \quad \frac{\partial q_i}{\partial p'_j} = -\frac{\partial q'_j}{\partial p_i}\,, \quad \frac{\partial p_i}{\partial q'_j} = -\frac{\partial p'_j}{\partial q_i}\,. \qquad \text{(B.31)}
$$

Taken together, (B.31) and (B.30) imply that the flow conserves the $\{p, q\}$ Poisson brackets

$$
\begin{aligned}
\{q_i, q_j\} &= \frac{\partial q_i}{\partial p'_k}\frac{\partial q_j}{\partial q'_k} - \frac{\partial q_j}{\partial p'_k}\frac{\partial q_i}{\partial q'_k} = 0 \\
\{p_i, p_j\} &= 0\,, \qquad \{p_i, q_j\} = \delta_{ij}\,,
\end{aligned}
\qquad \text{(B.32)}
$$

i.e., the transformations induced by a Hamiltonian flow are *canonical*, preserving the form of the equations of motion. The first two relations are symmetric under i, j interchange and yield $D(D-1)/2$ constraints each; the last relation yields D^2 constraints. Hence only $(2D)^2 - 2D(D-1)/2 - D^2 = 2D^2 + D$ elements of M are linearly independent, as it behooves group elements of the symplectic group $Sp(2D)$.

We have now succeeded in making the full set of constraints explicit - as we shall see in appendix C, this will enable us to implement dynamics in such a way that the symplectic invariance will be automatically preserved.

B.4 Monodromy matrix for Hamiltonian flows

(G. Tanner)

It is not the fundamental matrix of the flow, but the *monodromy* matrix, which enters the trace formula. This matrix gives the time dependence of a displacement perpendicular to the flow on the energy manifold. Indeed, we discover some trivial parts in the fundamental matrix M. An initial displacement in the direction of the flow $x = \omega \nabla H(x)$ transfers according to $\delta x(t) = x_t(t)\delta t$ with δt time independent. The projection of any displacement on δx on $\nabla H(x)$ is constant, i.e., $\nabla H(x(t))\delta x(t) = \delta E$. We get the equations of motion for the monodromy matrix directly choosing a suitable

local coordinate system on the orbit $x(t)$ in form of the (non singular) transformation $\mathbf{U}(x(t))$:

$$
\tilde{M}(x(t)) = \mathbf{U}^{-1}(x(t))\, M(x(t))\, \mathbf{U}(x(0)) \qquad \text{(B.33)}
$$

These lead to

$$
\begin{aligned}
&& \dot{\tilde{M}} &= \tilde{\mathbf{L}}\,\tilde{M} \\
\text{with} && \tilde{\mathbf{L}} &= \mathbf{U}^{-1}(\mathbf{L}\mathbf{U} - \dot{\mathbf{U}})
\end{aligned}
\qquad \text{(B.34)}
$$

Note that the properties a) – c) are only fulfilled for $\tilde{M}$ and $\tilde{\mathbf{L}}$, if $\mathbf{U}$ itself is symplectic.
Choosing $x_E = \nabla H(t)/|\nabla H(t)|^2$ and x_t as local coordinates uncovers the two trivial eigenvalues 1 of the transformed matrix in (B.33) at any time t. Setting $\mathbf{U} = (x_t^T, x_E^T, x_1^T, \ldots, x_{2d-2}^T)$ gives

$$
\tilde{M} = \begin{pmatrix} 1 & * & * & \ldots & * \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & * & & & \\ \vdots & \vdots & & \mathbf{m} & \\ 0 & * & & & \end{pmatrix}; \qquad \tilde{\mathbf{L}} = \begin{pmatrix} 0 & * & * & \ldots & * \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & * & & & \\ \vdots & \vdots & & \mathbf{l} & \\ 0 & * & & & \end{pmatrix}, \qquad \text{(B.35)}
$$

The matrix $\mathbf{m}$ is now the monodromy matrix and the equation of motion are given by

$$
\dot{\mathbf{m}} = \mathbf{l}\,\mathbf{m}. \qquad \text{(B.36)}
$$

The vectors $x_1, \ldots, x_{2d-2}$ must span the space perpendicular to the flow on the energy manifold.

For a system with two degrees of freedom, the matrix $\mathbf{U}(\mathbf{t})$ can be written down explicitly, i.e.,

$$
\mathbf{U}(t) = (x_t, x_1, x_E, x_2) = \begin{pmatrix} \dot{x} & -\dot{y} & -\dot{u}/q^2 & -\dot{v}/q^2 \\ \dot{y} & \dot{x} & -\dot{v}/q^2 & \dot{u}/q^2 \\ \dot{u} & \dot{v} & \dot{x}/q^2 & -\dot{y}/q^2 \\ \dot{v} & -\dot{u} & \dot{y}/q^2 & \dot{x}/q^2 \end{pmatrix} \qquad \text{(B.37)}
$$

with $x^T = (x, y; u, v)$ and $q = |\nabla H| = |\dot{x}|$. The matrix $\mathbf{U}$ is non singular and symplectic at every phase space point x (except the equilibrium points $\dot{x} = 0$). The matrix elements for $\mathbf{l}$ are given (B.39). One distinguishes 4 classes of eigenvalues of $\mathbf{m}$.

- *stable* or *elliptic*, if $\Lambda = e^{\pm i\pi\nu}$ and $\nu \in]0, 1[$.

- *marginal*, if $\Lambda = \pm 1$.
- *hyperbolic*, *inverse hyperbolic*, if $\Lambda = e^{\pm\lambda}$, $\Lambda = -e^{\pm\lambda}$; $\lambda > 0$ is called the Lyapunov exponent of the periodic orbit.
- *loxodromic*, if $\Lambda = e^{\pm u \pm i\Psi}$ with u and Ψ real. This is the most general case possible only in systems with 3 or more degree of freedoms.

For 2 degrees of freedom, i.e., $\mathbf{m}$ is a $[2\times 2]$ matrix, the eigenvalues are determined by

$$\lambda = \frac{\mathrm{Tr}(\mathbf{m}) \pm \sqrt{\mathrm{Tr}(\mathbf{m})^2 - 4}}{2}, \tag{B.38}$$

i.e., $\mathrm{Tr}(\mathbf{m}) = 2$ separates stable and unstable behavior.

The $\mathbf{l}$ matrix elements for the local transformation (B.37) are

$$\begin{aligned}
\tilde{\mathbf{l}}_{11} &= \frac{1}{q}[(h_x^2 - h_y^2 - h_u^2 + h_v^2)(h_{xu} - h_{yv}) + 2(h_x h_y - h_u h_v)(h_{xv} + h_{yu}) \\
&\quad -(h_x h_u + h_y h_v)(h_{xx} + h_{yy} - h_{uu} - h_{vv})] \\
\tilde{\mathbf{l}}_{12} &= \frac{1}{q^2}[(h_x^2 + h_v^2)(h_{yy} + h_{uu}) + (h_y^2 + h_u^2)(h_{xx} + h_{vv}) \\
&\quad -2(h_x h_u + h_y h_v)(h_{xu} + h_{yv}) - 2(h_x h_y - h_u h_v)(h_{xy} - h_{uv})] \\
\tilde{\mathbf{l}}_{21} &= -(h_x^2 + h_y^2)(h_{uu} + h_{vv}) - (h_u^2 + h_v^2)(h_{xx} + h_{yy}) \\
&\quad +2(h_x h_u - h_y h_v)(h_{xu} - h_{yv}) + 2(h_x h_v + h_y h_u)(h_{xv} + h_{yu}) \\
\tilde{\mathbf{l}}_{22} &= -\tilde{\mathbf{l}}_{11},
\end{aligned} \tag{B.39}$$

with h_i, h_{ij} is the derivative of the Hamiltonian H with respect to the phase space coordinates and $q = |\nabla H|^2$.