Appendix H

Discrete symmetries of dynamics

Basic group-theoretic notions are recapitulated here: groups, irreducible representations, invariants. Our notation follows birdtracks.eu.

The key result is the construction of projection operators from invariant matrices. The basic idea is simple: a hermitian matrix can be diagonalized. If this matrix is an invariant matrix, it decomposes the reps of the group into direct sums of lower-dimensional reps. Most of computations to follow implement the spectral decomposition

$$\mathbf{M} = \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \cdots + \lambda_r \mathbf{P}_r \,,$$

which associates with each distinct root λ_i of invariant matrix $\mathbf{M}$ a projection operator (H.17):

$$\mathbf{P}_i = \prod_{j \neq i} \frac{\mathbf{M} - \lambda_j \mathbf{1}}{\lambda_i - \lambda_j} \,.$$

Sects. H.3 and H.4 develop Fourier analysis as an application of the general theory of invariance groups and their representations.

H.1 Preliminaries and definitions

(A. Wirzba and P. Cvitanović)

We define *group*, *representation*, *symmetry of a dynamical system*, and *invariance.*

Group axioms. A group G is a set of elements $g_1, g_2, g_3, \ldots$ for which *composition* or *group multiplication* $g_2 \circ g_1$ (which we often abbreviate as $g_2 g_1$) of any two elements satisfies the following conditions:

1. If $g_1, g_2 \in G$, then $g_2 \circ g_1 \in G$.
2. The group multiplication is associative: $g_3 \circ (g_2 \circ g_1) = (g_3 \circ g_2) \circ g_1$.
3. The group G contains *identity* element e such that $g \circ e = e \circ g = g$ for every element $g \in G$.
4. For every element $g \in G$, there exists a unique $h == g^{-1} \in G$ such that $h \circ g = g \circ h = e$.

A *finite* group is a group with a finite number of elements

$$G = \{e, g_2, \ldots, g_{|G|}\} \,,$$

where $|G|$, the number of elements, is the *order* of the group.

Example H.1 Finite groups: *Some finite groups that frequently arise in applications:*

- C_n *(also denoted Z_n): the* cyclic group *of order n.*
- D_n*: the* dihedral group *of order $2n$, rotations and reflections in plane that preserve a regular n-gon.*
- S_n*: the* symmetric group *of all permutations of n symbols, order $n!$.*

Example H.2 Lie groups: *Some compact continuous groups that arise in dynamical systems applications:*

- S^1 *(also denoted T^1):* circle group *of dimension 1.*
- $T_m = S^1 \times S^1 \cdots \times S^1$: m-torus, *of dimension m.*
- $SO(2)$*: rotations in the plane, dimension 1. Isomorphic to S^1.*
- $O(2) = SO(2) \times D_1$*: group of rotations and reflections in the plane, of dimension 1.*
- $U(1)$*: group of phase rotations in the complex plane, of dimension 1. Isomorphic to $SO(2)$.*
- $SO(3)$: rotation group *of dimension 3.*
- $SU(2)$: unitary group *of dimension 3. Isomorphic to $SO(3)$.*
- $GL(n)$: general linear group *of invertible matrix transformations, dimension n^2.*
- $SO(n)$: special orthogonal group *of dimension $n(n-1)/2$.*
- $O(n) = SO(n) \times D_1$: orthogonal group *of dimension $n(n-1)/2$.*
- $Sp(n)$: symplectic group *of dimension $n(n+1)/2$.*
- $SU(n)$: special unitary group *of dimension $n^2 - 1$.*

Example H.3 Cyclic and dihedral groups: *The cyclic group $C_n \subset SO(2)$ of order n is generated by one element. For example, this element can be rotation through $2\pi/n$. The dihedral group $D_n \subset O(2)$, $n > 2$, can be generated by two elements one at least of which must reverse orientation. For example, take σ corresponding to reflection in the x-axis. $\sigma^2 = e$; such operation σ is called an* involution. *C to rotation through $2\pi/n$, then $D_n = \langle\sigma, C\rangle$, and the defining relations are $\sigma^2 = C^n = e$, $(C\sigma)^2 = e$.*

Groups are defined and classified as abstract objects by their multiplication tables (for finite groups) or Lie algebras (for Lie groups). What concerns us in applications is their *action* as groups of transformations on a given space, usually a vector space (see appendix B.1), but sometimes an affine space, or a more general manifold $\mathcal{M}$.

Repeated index summation. Throughout this text, the repeated pairs of upper/lower indices are always summed over

$$G_a{}^b x_b \equiv \sum_{b=1}^{n} G_a{}^b x_b \,, \tag{H.1}$$

unless explicitly stated otherwise.

General linear transformations. Let $GL(n,\mathbb{F})$ be the group of general linear transformations,

$$GL(n,\mathbb{F}) = \{\mathbf{g} : \mathbb{F}^n \to \mathbb{F}^n \,|\, \det(\mathbf{g}) \neq 0\} \,. \tag{H.2}$$

Under $GL(n,\mathbb{F})$ a basis set of V is mapped into another basis set by multiplication with a $[n\times n]$ matrix $\mathbf{g}$ with entries in field $\mathbb{F}$ ($\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$),

$$\mathbf{e}'^{\,a} = \mathbf{e}^b (\mathbf{g}^{-1})_b{}^a \,.$$

As the vector $\mathbf{x}$ is what it is, regardless of a particular choice of basis, under this transformation its coordinates must transform as

$$x'_a = g_a{}^b x_b \,.$$

Standard rep. We shall refer to the set of $[n\times n]$ matrices $\mathbf{g}$ as a *standard rep* of $GL(n,\mathbb{F})$, and the space of all n-tuples $(x_1, x_2, \ldots, x_n)^T$, $x_i \in \mathbb{F}$ on which these matrices act as the *standard representation space* V.

Under a general linear transformation $\mathbf{g} \in GL(n,\mathbb{F})$, the row of basis vectors transforms by right multiplication as $\mathbf{e}' = \mathbf{e}\,\mathbf{g}^{-1}$, and the column of

x_a's transforms by left multiplication as $x' = \mathbf{g}x$. Under left multiplication the column (row transposed) of basis vectors $\mathbf{e}^T$ transforms as $\mathbf{e}'^T = \mathbf{g}^\dagger \mathbf{e}^T$, where the *dual rep* $\mathbf{g}^\dagger = (\mathbf{g}^{-1})^T$ is the transpose of the inverse of $\mathbf{g}$. This observation motivates introduction of a *dual* representation space $\bar{V}$, the space on which $GL(n,\mathbb{F})$ acts via the dual rep $\mathbf{g}^\dagger$.

Dual space. If V is a vector representation space, then the *dual space* $\bar{V}$ is the set of all linear forms on V over the field $\mathbb{F}$.

If $\{\mathbf{e}^{(1)}, \cdots, \mathbf{e}^{(d)}\}$ is a (right) basis of V, then $\bar{V}$ is spanned by the *dual basis* (left basis) $\{\mathbf{e}_{(1)}, \cdots, \mathbf{e}_{(d)}\}$, the set of n linear forms $\mathbf{e}_{(j)}$ such that

$$\mathbf{e}_{(i)} \cdot \mathbf{e}^{(j)} = \delta_i^j \,,$$

where δ_a^b is the Kronecker symbol, $\delta_a^b = 1$ if $a = b$, and zero otherwise. The components of dual representation space vectors will here be distinguished by upper indices

$$(y^1, y^2, \ldots, y^n) \,. \tag{H.3}$$

They transform under $GL(n,\mathbb{F})$ as

$$y'^a = (\mathbf{g}^\dagger)_b{}^a y^b \,. \tag{H.4}$$

For $GL(n,\mathbb{F})$ no complex conjugation is implied by the † notation; that interpretation applies only to unitary subgroups of $GL(n,\mathbb{C})$. $\mathbf{g}$ can be distinguished from $\mathbf{g}^\dagger$ by meticulously keeping track of the relative ordering of the indices,

$$g_a^b \to g_a{}^b \,, \qquad (\mathbf{g}^\dagger)_a^b \to g^b{}_a \,. \tag{H.5}$$

Defining space, dual space. In what follows V will always denote the *defining* n-dimensional complex vector representation space, that is to say the initial, "elementary multiplet" space within which we commence our deliberations. Along with the defining vector representation space V comes the *dual* n-dimensional vector representation space $\bar{V}$. We shall denote the corresponding element of $\bar{V}$ by raising the index, as in (H.3), so the components of defining space vectors, resp. dual vectors, are distinguished by lower, resp. upper indices:

$$\begin{aligned} x &= (x_1, x_2, \ldots, x_n) \,, \qquad \mathbf{x} \in V \\ \bar{x} &= (x^1, x^2, \ldots, x^n) \,, \qquad \bar{\mathbf{x}} \in \bar{V} \,. \end{aligned} \tag{H.6}$$

Defining rep. Let G be a group of transformations acting linearly on V, with the action of a group element $g \in G$ on a vector $x \in V$ given by an $[n \times n]$ matrix $\mathbf{g}$

$$x'_a = g_a{}^b x_b \qquad a, b = 1, 2, \ldots, n\,. \tag{H.7}$$

We shall refer to $g_a{}^b$ as the *defining rep* of the group G. The action of $g \in G$ on a vector $\bar{q} \in \bar{V}$ is given by the *dual rep* $[n \times n]$ matrix $\mathbf{g}^\dagger$:

$$x'^a = x^b (\mathbf{g}^\dagger)_b{}^a = g^a{}_b x^b\,. \tag{H.8}$$

In the applications considered here, the group G will almost always be assumed to be a subgroup of the *unitary group*, in which case $\mathbf{g}^{-1} = \mathbf{g}^\dagger$, and † indicates hermitian conjugation:

$$(\mathbf{g}^\dagger)_a{}^b = (g_b{}^a)^* = g^b{}_a\,. \tag{H.9}$$

Hermitian conjugation is effected by complex conjugation and index transposition: Complex conjugation interchanges upper and lower indices; transposition reverses their order. A matrix is *hermitian* if its elements satisfy

$$(\mathbf{M}^\dagger)^a_b = M^a_b\,. \tag{H.10}$$

For a hermitian matrix there is no need to keep track of the relative ordering of indices, as $M_b{}^a = (\mathbf{M}^\dagger)_b{}^a = M^a{}_b$.

Invariant vectors. The vector $q \in V$ is an *invariant vector* if for any transformation $g \in G$

$$q = \mathbf{g}q\,. \tag{H.11}$$

If a bilinear form $\mathbf{M}(\bar{x}, y) = x^a M_a{}^b y_b$ is invariant for all $g \in G$, the matrix

$$M_a{}^b = g_a{}^c g^b{}_d M_c{}^d \tag{H.12}$$

is an *invariant matrix*. Multiplying with $g_b{}^e$ and using the unitary condition (H.9), we find that the invariant matrices *commute* with all transformations $g \in G$:

$$[\mathbf{g}, \mathbf{M}] = 0\,. \tag{H.13}$$

Invariants. We shall refer to an invariant relation between p vectors in V and q vectors in $\bar{V}$, which can be written as a homogeneous polynomial in terms of vector components, such as

$$H(x, y, \bar{z}, \bar{r}, \bar{s}) = h^{ab}{}_{cde} x_b y_a s^e r^d z^c\,, \tag{H.14}$$

as an *invariant* in $V^q \otimes \bar{V}^p$ (repeated indices, as always, summed over). In this example, the coefficients $h^{ab}{}_{cde}$ are components of invariant tensor $h \in V^3 \otimes \bar{V}^2$.

Matrix group on vector space. We will now apply these abstract group definitions to the set of $[d \times d]$-dimensional non-singular matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots \in GL(d)$ acting in a d-dimensional vector space $V \in \mathbb{R}^d$. The product of matrices $\mathbf{A}$ and $\mathbf{B}$ gives the matrix $\mathbf{C}$,

$$\mathbf{C}x = \mathbf{B}(\mathbf{A}x) = (\mathbf{BA})x \in V, \qquad \forall x \in V.$$

The identity of the group is the unit matrix $\mathbf{1}$ which leaves all vectors in V unchanged. Every matrix in the group has a unique inverse.

Matrix representation of a group. Let us now map the abstract group G *homeomorphically* on a group of matrices $\mathsf{D}(G)$ acting on the vector space V, i.e., in such a way that the group properties, especially the group multiplication, are preserved:

1. Any $g \in G$ is mapped to a matrix $\mathbf{D}(g) \in \mathsf{D}(G)$.
2. The group product $g_2 \circ g_1 \in G$ is mapped onto the matrix product $\mathbf{D}(g_2 \circ g_1) = \mathbf{D}(g_2)\mathbf{D}(g_1)$.
3. The associativity is preserved: $\mathbf{D}(g_3 \circ (g_2 \circ g_1)) = \mathbf{D}(g_3)(\mathbf{D}(g_2)\mathbf{D}(g_1)) = (\mathbf{D}(g_3)(\mathbf{D}(g_2))\mathbf{D}(g_1)$.
4. The identity element $e \in G$ is mapped onto the unit matrix $\mathsf{D}(e) = \mathbf{1}$ and the inverse element $g^{-1} \in G$ is mapped onto the inverse matrix $\mathbf{D}(g^{-1}) = [\mathbf{D}(g)]^{-1} \equiv \mathbf{D}^{-1}(g)$.

We call this matrix group $\mathsf{D}(G)$ a linear or matrix *representation* of the group G in the *representation space* V. We emphasize here *'linear'* in order to distinguish the matrix representations from other representations that do not have to be linear, in general. Throughout this appendix we only consider linear representations.

If the dimensionality of V is d, we say the representation is an *d-dimensional representation*. We will often abbreviate the notation by writing matrices $\mathbf{D}(g) \in \mathsf{D}(G)$ as $\mathbf{g}$, i.e., $x' = \mathbf{g}x$ corresponds to the matrix operation $x'_i = \sum_{j=1}^d \mathbf{D}(g)_{ij} x_j$.

Character of a representation. The character of $\chi_\alpha(g)$ of a d-dimensional representation $\mathbf{D}(g)$ of the group element $g \in G$ is defined as trace

$$\chi_\alpha(g) = \mathrm{tr}\,\mathbf{D}(g) = \sum_{i=1}^{d} \mathbf{D}_{ii}(g)\,.$$

Note that $\chi(e) = d$, since $\mathbf{D}_{ij}(e) = \delta_{ij}$ for $1 \le i, j \le d$.

Faithful representations, factor group. If the mapping G on $\mathsf{D}(G)$ is an isomorphism, the representation is said to be *faithful*. In this case the order of the group of matrices $D(G)$ is equal to the order $|G|$ of the group. In general, however, there will be several elements $h \in G$ that will be mapped on the unit matrix $\mathbf{D}(h) = \mathbf{1}$. This property can be used to define a subgroup $H \subset G$ of the group G consisting of all elements $h \in G$ that are mapped to the unit matrix of a given representation. Then the representation is a faithful representation of the *factor group* G/H.

Equivalent representations, equivalence classes. A representation of a group is by no means unique. If the basis in the d-dimensional vector space V is changed, the matrices $\mathbf{D}(g)$ have to be replaced by their transformations $\mathbf{D}'(g)$, with the new matrices $\mathbf{D}'(g)$ and the old matrices $\mathbf{D}(g)$ are related by an equivalence transformation through a non-singular matrix $\mathbf{C}$

$$\mathbf{D}'(g) = \mathbf{C}\,\mathbf{D}(g)\,\mathbf{C}^{-1}\,.$$

The group of matrices $\mathbf{D}'(g)$ form a representation $\mathsf{D}'(G)$ equivalent to the representation $\mathsf{D}(G)$ of the group G. The equivalent representations have the same structure, although the matrices look different. Because of the cylic nature of the trace the character of equivalent representations is the same

$$\chi(g) = \sum_{i=1}^{n} \mathbf{D}'_{ii}(g) = \mathrm{tr}\,\mathbf{D}'(g) = \mathrm{tr}\left(\mathbf{C}\mathbf{D}(g)\mathbf{C}^{-1}\right)\,.$$

Regular representation of a finite group. The *regular* representation of a group is a special representation that is defined as follows: Combine the elements of a finite group into a vector $\{g_1, g_2, \ldots, g_{|G|}\}$. Multiplication by any element g_ν permutes $\{g_1, g_2, \ldots, g_{|G|}\}$ entries. We can represent the element g_ν by the permutation it induces on the components of vector $\{g_1, g_2, \ldots, g_{|G|}\}$. Thus for $i, j = 1, \ldots, |G|$, we define the *regular representation*

$$\mathbf{D}_{ij}(g_\nu) = \begin{cases} \delta_{jl_i} & \text{if } g_\nu g_i = g_{l_i} \text{ with } l_i = 1, \ldots, |G|\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

In the regular representation the diagonal elements of all matrices are zero except for the identity element $g_\nu = e$ with $g_\nu g_i = g_i$. So in the regular representation the character is given by

$$\chi(g) = \begin{cases} |G| & \text{for } \; g = e\,, \\ 0 & \text{for } \; g \ne e\,. \end{cases}$$

H.2 Invariants and reducibility

What follows is a bit dry, so we start with a motivational quote from Hermann Weyl on the "so-called first main theorem of invariant theory":

"*All invariants are expressible in terms of a finite number among them.* We cannot claim its validity for every group G; rather, it will be our chief task to investigate for each particular group whether a finite integrity basis exists or not; the answer, to be sure, will turn out affirmative in the most important cases."

It is easy to show that any rep of a finite group can be brought to unitary form, and the same is true of all compact Lie groups. Hence, in what follows, we specialize to unitary and hermitian matrices.

H.2.1 Projection operators

For $\mathbf{M}$ a hermitian matrix, there exists a diagonalizing unitary matrix $\mathbf{C}$ such that

$$\mathbf{C}\mathbf{M}\mathbf{C}^\dagger = \left(\begin{array}{c|c|c} \begin{array}{ccc} \lambda_1 & \ldots & 0 \\ & \ddots & \\ 0 & \ldots & \lambda_1 \end{array} & 0 & 0 \\ \hline 0 & \begin{array}{cccc} \lambda_2 & 0 & \ldots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \ldots & \lambda_2 \end{array} & 0 \\ \hline 0 & 0 & \begin{array}{cc} \lambda_3 & \ldots \\ \vdots & \ddots \end{array} \end{array}\right)\,. \tag{H.15}$$

Here $\lambda_i \ne \lambda_j$ are the r distinct roots of the minimal *characteristic* (or *secular*) polynomial

$$\prod_{i=1}^{r} (\mathbf{M} - \lambda_i \mathbf{1}) = 0\,. \tag{H.16}$$

In the matrix $\mathbf{C}(\mathbf{M} - \lambda_2 \mathbf{1})\mathbf{C}^\dagger$ the eigenvalues corresponding to λ_2 are replaced by zeroes:

$$\begin{pmatrix} \begin{array}{cc|} \lambda_1 - \lambda_2 & \\ & \lambda_1 - \lambda_2 \\ \hline \end{array} & & \\ & \boxed{\begin{array}{ccc} 0 & & \\ & \ddots & \\ & & 0 \end{array}} & \\ & & \begin{array}{|ccc} \hline \lambda_3 - \lambda_2 & & \\ & \lambda_3 - \lambda_2 & \\ & & \ddots \end{array} \end{pmatrix},$$

and so on, so the product over all factors $(\mathbf{M} - \lambda_2 \mathbf{1})(\mathbf{M} - \lambda_3 \mathbf{1}) \ldots$, with exception of the $(\mathbf{M} - \lambda_1 \mathbf{1})$ factor, has nonzero entries only in the subspace associated with λ_1:

$$\mathbf{C} \prod_{j \neq 1} (\mathbf{M} - \lambda_j \mathbf{1}) \mathbf{C}^\dagger = \prod_{j \neq 1} (\lambda_1 - \lambda_j) \begin{pmatrix} \begin{array}{ccc|} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline \end{array} & 0 \\ 0 & \begin{array}{|cccc} \hline 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{array} \end{pmatrix}.$$

Thus we can associate with each distinct root λ_i a *projection operator* $\mathbf{P}_i$,

$$\mathbf{P}_i = \prod_{j \neq i} \frac{\mathbf{M} - \lambda_j \mathbf{1}}{\lambda_i - \lambda_j}, \tag{H.17}$$

which acts as identity on the ith subspace, and zero elsewhere. For example, the projection operator onto the λ_1 subspace is

$$\mathbf{P}_1 = \mathbf{C}^\dagger \begin{pmatrix} \begin{array}{ccc|} 1 & & \\ & \ddots & \\ & & 1 \\ \hline \end{array} & \\ & \begin{array}{|cccc} \hline 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{array} \end{pmatrix} \mathbf{C}. \tag{H.18}$$

The diagonalization matrix $\mathbf{C}$ is deployed in the above only as a pedagogical device. The whole point of the projector operator formalism is that we *never* need to carry such explicit diagonalization; all we need are whatever invariant matrices $\mathbf{M}$ we find convenient, the algebraic relations they satisfy, and orthonormality and completeness of $\mathbf{P}_i$: The matrices $\mathbf{P}_i$ are *orthogonal*

$$\mathbf{P}_i \mathbf{P}_j = \delta_{ij} \mathbf{P}_j, \qquad (\text{no sum on } j), \tag{H.19}$$

and satisfy the *completeness relation*

$$\sum_{i=1}^{r} \mathbf{P}_i = \mathbf{1}. \tag{H.20}$$

As $\mathrm{tr}\,(\mathbf{C}\mathbf{P}_i\mathbf{C}^\dagger) = \mathrm{tr}\,\mathbf{P}_i$, the dimension of the ith subspace is given by

$$d_i = \mathrm{tr}\,\mathbf{P}_i. \tag{H.21}$$

It follows from the characteristic equation (H.16) and the form of the projection operator (H.17) that λ_i is the eigenvalue of $\mathbf{M}$ on $\mathbf{P}_i$ subspace:

$$\mathbf{M}\mathbf{P}_i = \lambda_i \mathbf{P}_i, \qquad (\text{no sum on } i). \tag{H.22}$$

Hence, any matrix polynomial $f(\mathbf{M})$ takes the scalar value $f(\lambda_i)$ on the $\mathbf{P}_i$ subspace

$$f(\mathbf{M})\mathbf{P}_i = f(\lambda_i)\mathbf{P}_i. \tag{H.23}$$

This, of course, is the reason why one wants to work with irreducible reps: they reduce matrices and "operators" to pure numbers.

H.2.2 Irreducible representations

Suppose there exist several linearly independent invariant $[d \times d]$ hermitian matrices $\mathbf{M}_1, \mathbf{M}_2, \ldots$, and that we have used $\mathbf{M}_1$ to decompose the d-dimensional vector space $V = V_1 \oplus V_2 \oplus \cdots$. Can $\mathbf{M}_2, \mathbf{M}_3, \ldots$ be used to further decompose V_i? Further decomposition is possible if, and only if, the invariant matrices commute:

$$[\mathbf{M}_1, \mathbf{M}_2] = 0, \tag{H.24}$$

or, equivalently, if projection operators $\mathbf{P}_j$ constructed from $\mathbf{M}_2$ commute with projection operators $\mathbf{P}_i$ constructed from $\mathbf{M}_1$,

$$\mathbf{P}_i \mathbf{P}_j = \mathbf{P}_j \mathbf{P}_i. \tag{H.25}$$

Usually the simplest choices of independent invariant matrices do not commute. In that case, the projection operators $\mathbf{P}_i$ constructed from $\mathbf{M}_1$ can be used to project commuting pieces of $\mathbf{M}_2$:

$$\mathbf{M}_2^{(i)} = \mathbf{P}_i \mathbf{M}_2 \mathbf{P}_i, \qquad (\text{no sum on } i).$$

That $\mathbf{M}_2^{(i)}$ commutes with $\mathbf{M}_1$ follows from the orthogonality of $\mathbf{P}_i$:

$$[\mathbf{M}_2^{(i)}, \mathbf{M}_1] = \sum_j \lambda_j [\mathbf{M}_2^{(i)}, \mathbf{P}_j] = 0 \,. \tag{H.26}$$

Now the characteristic equation for $\mathbf{M}_2^{(i)}$ (if nontrivial) can be used to decompose V_i subspace.

An invariant matrix $\mathbf{M}$ induces a decomposition only if its diagonalized form (H.15) has more than one distinct eigenvalue; otherwise it is proportional to the unit matrix and commutes trivially with all group elements. A rep is said to be *irreducible* if all invariant matrices that can be constructed are proportional to the unit matrix.

According to (H.13), an invariant matrix $\mathbf{M}$ commutes with group transformations $[G, \mathbf{M}] = 0$. Projection operators (H.17) constructed from $\mathbf{M}$ are polynomials in $\mathbf{M}$, so they also commute with all $g \in \mathcal{G}$:

$$[G, \mathbf{P}_i] = 0 \tag{H.27}$$

Hence, a $[d \times d]$ matrix rep can be written as a direct sum of $[d_i \times d_i]$ matrix reps:

$$G = \mathbf{1}G\mathbf{1} = \sum_{i,j} \mathbf{P}_i G \mathbf{P}_j = \sum_i \mathbf{P}_i G \mathbf{P}_i = \sum_i G_i \,. \tag{H.28}$$

In the diagonalized rep (H.18), the matrix $\mathbf{g}$ has a block diagonal form:

$$\mathbf{C}\mathbf{g}\mathbf{C}^\dagger = \begin{bmatrix} \mathbf{g}_1 & 0 & 0 \\ 0 & \mathbf{g}_2 & 0 \\ 0 & 0 & \ddots \end{bmatrix} \,, \qquad \mathbf{g} = \sum_i \mathbf{C}^i \mathbf{g}_i \mathbf{C}_i \,. \tag{H.29}$$

The rep $\mathbf{g}_i$ acts only on the d_i-dimensional subspace V_i consisting of vectors $\mathbf{P}_i q$, $q \in V$. In this way an invariant $[d \times d]$ hermitian matrix $\mathbf{M}$ with r distinct eigenvalues induces a decomposition of a d-dimensional vector space V into a direct sum of d_i-dimensional vector subspaces V_i:

$$V \xrightarrow{\mathbf{M}} V_1 \oplus V_2 \oplus \ldots \oplus V_r \,. \tag{H.30}$$

H.3 Lattice derivatives

Consider a smooth function $\phi(x)$ evaluated on a finite d-dimensional lattice

$$\phi_\ell = \phi(x) \,, \qquad x = a\ell = \text{lattice point} \,, \quad \ell \in \mathbf{Z}^d \,, \tag{H.31}$$

where a is the lattice spacing and there are N^d points in all. A vector ϕ specifies a lattice configuration. Assume the lattice is hyper-cubic, and let $\hat{n}_\mu \in \{\hat{n}_1, \hat{n}_2, \cdots, \hat{n}_d\}$ be the unit lattice cell vectors pointing along the d positive directions, $|\hat{n}_\mu| = 1$. The *lattice partial derivative* is then

$$(\partial_\mu \phi)_\ell = \frac{\phi(x + a\hat{n}_\mu) - \phi(x)}{a} = \frac{\phi_{\ell+\hat{n}_\mu} - \phi_\ell}{a} \,.$$

Anything else with the correct $a \to 0$ limit would do, but this is the simplest choice. We can rewrite the derivative as a linear operator, by introducing the *hopping operator* (or "shift", or "step") in the direction μ

$$(\mathbf{h}_\mu)_{\ell j} = \delta_{\ell + \hat{n}_\mu, j} \,. \tag{H.32}$$

As $\mathbf{h}$ will play a central role in what follows, it pays to understand what it does, so we write it out for the 1-dimensional case in its full $[N \times N]$ matrix glory:

$$\mathbf{h} = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & & \ddots & \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{pmatrix} \,. \tag{H.33}$$

We will assume throughout that the lattice is *periodic* in each $\hat{n}_\mu$ direction; this is the easiest boundary condition to work with if we are interested in large lattices where surface effects are negligible.

Applied on the lattice configuration $\phi = (\phi_1, \phi_2, \cdots, \phi_N)$, the hopping operator shifts the lattice by one site, $\mathbf{h}\phi = (\phi_2, \phi_3, \cdots, \phi_N, \phi_1)$. Its transpose shifts the entries the other way, so the transpose is also the inverse

$$\mathbf{h}^{-1} = \mathbf{h}^T \,. \tag{H.34}$$

The lattice derivative can now be written as a multiplication by a matrix:

$$\partial_\mu \phi_\ell = \frac{1}{a} (\mathbf{h}_\mu - \mathbf{1})_{\ell j} \, \phi_j \,.$$

In the 1-dimensional case the $[N \times N]$ matrix representation of the lattice derivative is:

$$\partial = \frac{1}{a} \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & -1 \end{pmatrix} \,. \tag{H.35}$$

To belabor the obvious: On a finite lattice of N points a derivative is simply a finite $[N\times N]$ matrix. Continuum field theory is a world in which the lattice is so fine that it looks smooth to us. Whenever someone calls something an "operator", think "matrix". For finite-dimensional spaces a linear operator *is* a matrix; things get subtler for infinite-dimensional spaces.

H.3.1 Lattice Laplacian

In order to get rid of some of the lattice indices it is convenient to employ vector notation for the terms bilinear in ϕ, and keep the rest lumped into "interaction",

$$S[\phi] = -\frac{M^2}{2}\,\phi^T\cdot\phi - \frac{C}{2}\left[(\mathbf{h}_\mu - \mathbf{1})\,\phi\right]^T\cdot(\mathbf{h}_\mu-\mathbf{1})\,\phi + S_I[\phi]\,. \tag{H.36}$$

For example, for the discretized Landau Hamiltonian $M^2/2 = \beta m_0^2/2$, $C = \beta/a^2$, and the quartic term $S_I[\phi]$ is local site-by-site, $\gamma_{\ell_1\ell_2\ell_3\ell_4} = -4!\,\beta u\,\delta_{\ell_1\ell_2}\delta_{\ell_2\ell_3}\delta_{\ell_3\ell_4}$, so this general quartic coupling is a little bit of an overkill, but by the time we get to the Fourier-transformed theory, it will make sense as a momentum conserving vertex (H.59).

In the continuum integration by parts moves ∂_μ around; on a lattice this amounts to a matrix transposition

$$\left[(\mathbf{h}_\mu-\mathbf{1})\,\phi\right]^T\cdot\left[(\mathbf{h}_\mu-\mathbf{1})\,\phi\right] = \phi^T\cdot(\mathbf{h}_\mu^{-1}-\mathbf{1})\,(\mathbf{h}_\mu-\mathbf{1})\cdot\phi\,.$$

If you are wondering where the "integration by parts" minus sign is, it is there in discrete case at well. It comes from the identity $\partial^T = -\mathbf{h}^{-1}\partial$. The combination $\Delta = \mathbf{h}^{-1}\partial^2$

$$\Delta = -\frac{1}{a^2}\sum_{\mu=1}^{d}(\mathbf{h}_\mu^{-1}-\mathbf{1})\,(\mathbf{h}_\mu-\mathbf{1}) = -\frac{2}{a^2}\sum_{\mu=1}^{d}\left(\mathbf{1}-\frac{1}{2}(\mathbf{h}_\mu^{-1}+\mathbf{h}_\mu)\right) \tag{H.37}$$

is the *lattice Laplacian*. We shall show below that this Laplacian has the correct continuum limit. It is the simplest spatial derivative allowed for $x\to -x$ symmetric actions. In the 1-dimensional case the $[N\times N]$ matrix representation of the lattice Laplacian is:

$$\Delta = \frac{1}{a^2}\begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & 1 & & \ddots & \\ & & & & & 1 \\ 1 & & & & 1 & -2 \end{pmatrix}. \tag{H.38}$$

The lattice Laplacian measures the second variation of a field ϕ_ℓ across three neighboring sites. You can easily check that it does what the second derivative is supposed to do by applying it to a parabola restricted to the lattice, $\phi_\ell = \phi(\ell)$, where $\phi(\ell)$ is defined by the value of the continuum function $\phi(x) = x^2$ at the lattice point ℓ.

H.4 Periodic lattices

Our task now is to transform M into a form suitable to evaluation of Feynman diagrams. The theory we will develop in this section is applicable only to *translationally invariant* saddle-point configurations.

Consider the effect of a $\phi\to\mathbf{h}\phi$ translation on the action (**??**)

$$S[\mathbf{h}\phi] = -\frac{1}{2}\phi^T\cdot\mathbf{h}^T M^{-1}\mathbf{h}\cdot\phi - \frac{\beta g_0}{4!}\sum_{\ell=1}^{N^d}(\mathbf{h}\phi)_\ell^4\,.$$

As M^{-1} is constructed from $\mathbf{h}$ and its inverse, M^{-1} and $\mathbf{h}$ commute, and the bilinear term is $\mathbf{h}$ invariant. In the quartic term $\mathbf{h}$ permutes cyclically the terms in the sum, so the total action is translationally invariant

$$S[\mathbf{h}\phi] = S[\phi] = -\frac{1}{2}\phi^T\cdot M^{-1}\cdot\phi - \frac{\beta g_0}{4!}\sum_{\ell=1}^{N^d}\phi_\ell^4\,.$$

If a function (in this case, the action $S[\phi]$) defined on a vector space (in this case, the configuration ϕ) commutes with a linear operator $\mathbf{h}$, then the eigenvalues of $\mathbf{h}$ can be used to decompose the ϕ vector space into invariant subspaces. For a hyper-cubic lattice the translations in different directions commute, $\mathbf{h}_\mu\mathbf{h}_\nu = \mathbf{h}_\nu\mathbf{h}_\mu$, so it is sufficient to understand the spectrum of the 1-dimensional shift operator (H.33). To develop a feeling for how this reduction to invariant subspaces works in practice, let us continue in the humble spirit of sect. **??**, by expanding the scope of our deliberations to a lattice consisting of 2 points.

H.4.1 A 2-point lattice diagonalized

The action of the shift operator $\mathbf{h}$ (H.33) on a 2-point lattice $\phi = (\phi_1,\phi_2)$ is to permute the two lattice sites

$$\mathbf{h} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

As exchange repeated twice brings us back to the original configuration, $\mathbf{h}^2 = \mathbf{1}$, and the characteristic polynomial of $\mathbf{h}$ is

$$(\mathbf{h}+1)(\mathbf{h}-1) = 0\,,$$

with eigenvalues $\lambda_0 = 1, \lambda_1 = -1$. Construct now the symmetrization, antisymmetrization projection operators

$$P_0 = \frac{\mathbf{h}-\lambda_1 \mathbf{1}}{\lambda_0-\lambda_1} = \frac{1}{2}(\mathbf{1}+\mathbf{h}) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{H.39}$$

$$P_1 = \frac{\mathbf{h}-\mathbf{1}}{-1-1} = \frac{1}{2}(\mathbf{1}-\mathbf{h}) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{H.40}$$

Noting that $P_0 + P_1 = \mathbf{1}$, we can project the lattice configuration ϕ onto the two eigenvectors of $\mathbf{h}$:

$$\begin{aligned} \phi &= \mathbf{1}\,\phi = P_0\cdot\phi + P_1\cdot\phi\,, \\ \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} &= \frac{(\phi_1+\phi_2)}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{(\phi_1-\phi_2)}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} && \text{(H.41)} \\ &= \tilde\phi_0\hat n_0 + \tilde\phi_1 \hat n_1\,. && \text{(H.42)} \end{aligned}$$

As $P_0P_1 = 0$, the symmetric and the antisymmetric configurations transform separately under any linear transformation constructed from $\mathbf{h}$ and its powers.

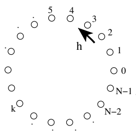
In this way the characteristic equation $\mathbf{h}^2 = \mathbf{1}$ enables us to reduce the 2-dimenional lattice configuration to two 1-dimensional ones, on which the value of the shift operator (shift matrix) $\mathbf{h}$ is a number, $\lambda \in \{1,-1\}$, and the eigenvectors are $\hat n_0 = \frac{1}{\sqrt{2}}(1,1)$, $\hat n_1 = \frac{1}{\sqrt{2}}(1,-1)$. We have inserted $\sqrt{2}$ factors only for convenience, in order that the eigenvectors be normalized unit vectors. As we shall now see, $(\tilde\phi_0, \tilde\phi_1)$ is the 2-site periodic lattice discrete Fourier transform of the field (ϕ_1, ϕ_2).

H.5 Discrete Fourier transforms

Now let us generalize this reduction to a 1-dimensional periodic lattice with N sites.

Each application of $\mathbf{h}$ translates the lattice one step; in N steps the lattice is back in the original configuration

$$\mathbf{h}^N = \mathbf{1}$$



,

so the eigenvalues of $\mathbf{h}$ are the N distinct N-th roots of unity

$$\mathbf{h}^N - \mathbf{1} = \prod_{k=0}^{N-1}(\mathbf{h}-\omega^k\mathbf{1}) = 0\,, \qquad \omega = e^{i\frac{2\pi}{N}}\,. \tag{H.43}$$

As the eigenvalues are all distinct and N in number, the space is decomposed into N one-dimensional subspaces. The general theory (expounded in appendix H.2) associates with the k-th eigenvalue of $\mathbf{h}$ a projection operator that projects a configuration ϕ onto k-th eigenvector of $\mathbf{h}$,

$$P_k = \prod_{j\neq k}\frac{\mathbf{h}-\lambda_j\mathbf{1}}{\lambda_k-\lambda_j}\,. \tag{H.44}$$

A factor $(\mathbf{h}-\lambda_j\mathbf{1})$ kills the j-th eigenvector φ_j component of an arbitrary vector in expansion $\phi = \cdots + \tilde\phi_j\varphi_j + \cdots$. The above product kills everything but the eigendirection φ_k, and the factor $\prod_{j\neq k}(\lambda_k-\lambda_j)$ ensures that P_k is normalized as a projection operator. The set of the projection operators is complete

$$\sum_k P_k = \mathbf{1} \tag{H.45}$$

and orthonormal

$$P_kP_j = \delta_{kj}P_k \qquad (\text{no sum on } k)\,. \tag{H.46}$$

Constructing explicit eigenvectors is usually not a the best way to fritter one's youth away, as choice of basis is largely arbitrary, and all of the content of the theory is in projection operators [**?**]. However, in case at hand the eigenvectors are so simple that we can forget the general theory, and construct the solutions of the eigenvalue condition

$$\mathbf{h}\,\varphi_k = \omega^k\varphi_k \tag{H.47}$$

by hand:

$$\frac{1}{\sqrt{N}}\begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & & \ddots & \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{pmatrix}\begin{pmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \omega^{3k} \\ \vdots \\ \omega^{(N-1)k} \end{pmatrix} = \omega^k\frac{1}{\sqrt{N}}\begin{pmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \omega^{3k} \\ \vdots \\ \omega^{(N-1)k} \end{pmatrix}$$

The $1/\sqrt{N}$ factor is chosen in order that φ_k be normalized unit vectors

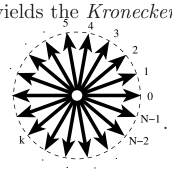
$$\begin{aligned} \varphi_k^\dagger \cdot \varphi_k &= \frac{1}{N}\sum_{k=0}^{N-1} 1 \;=\; 1\,, \qquad \text{(no sum on } k) \\ \varphi_k^\dagger &= \frac{1}{\sqrt{N}}\left(1, \omega^{-k}, \omega^{-2k}, \cdots, \omega^{-(N-1)k}\right)\,. \end{aligned} \tag{H.48}$$

The eigenvectors are orthonormal

$$\varphi_k^\dagger \cdot \varphi_j = \delta_{kj}\,, \tag{H.49}$$

as the explicit evaluation of $\varphi_k^\dagger \cdot \varphi_j$ yields the *Kronecker delta function for a periodic lattice*

$$\delta_{kj} = \frac{1}{N}\sum_{\ell=0}^{N-1} e^{i\frac{2\pi}{N}(k-j)\ell}$$



$$. \tag{H.50}$$

The sum is over the N unit vectors pointing at a uniform distribution of points on the complex unit circle; they cancel each other unless $k = j$ (mod N), in which case each term in the sum equals 1.

The projection operators can be expressed in terms of the eigenvectors (H.47), (H.48) as

$$(P_k)_{\ell\ell'} = (\varphi_k)_\ell(\varphi_k^\dagger)_{\ell'} = \frac{1}{N} e^{i\frac{2\pi}{N}(\ell-\ell')k}\,, \qquad \text{(no sum on } k)\,. \tag{H.51}$$

The completeness (H.45) follows from (H.50), and the orthonormality (H.46) from (H.49).

$\tilde{\phi}_k$, the projection of the ϕ configuration on the k-th subspace is given by

$$\begin{aligned} (P_k \cdot \phi)_\ell &= \tilde{\phi}_k\,(\varphi_k)_\ell\,, \qquad \text{(no sum on } k) \\ \tilde{\phi}_k &= \varphi_k^\dagger \cdot \phi \;=\; \frac{1}{\sqrt{N}}\sum_{\ell=0}^{N-1} e^{-i\frac{2\pi}{N}k\ell}\phi_\ell \end{aligned} \tag{H.52}$$

We recognize $\tilde{\phi}_k$ as the *discrete Fourier transform* of ϕ_ℓ. Hopefully rediscovering it this way helps you a little toward understanding why Fourier transforms are full of $e^{ix\cdot p}$ factors (they are eigenvalues of the generator of translations) and when are they the natural set of basis functions (only if the theory is translationally invariant).

H.5.1 Fourier transform of the propagator

Now insert the identity $\sum P_k = \mathbf{1}$ wherever profitable:

$$\mathbf{M} = \mathbf{1}\mathbf{M}\mathbf{1} = \sum_{kk'} P_k \mathbf{M} P_{k'} = \sum_{kk'} \varphi_k(\varphi_k^\dagger \cdot \mathbf{M} \cdot \varphi_{k'})\varphi_{k'}^\dagger\,.$$

The matrix

$$\tilde{M}_{kk'} = (\varphi_k^\dagger \cdot \mathbf{M} \cdot \varphi_{k'}) \tag{H.53}$$

is the Fourier space representation of $\mathbf{M}$. No need to stop here - the terms in the action (**??**) that couple 3, 4, $\cdots$ fields also have the Fourier space representations

$$\begin{aligned} \gamma_{\ell_1\ell_2\cdots\ell_n}\,\phi_{\ell_1}\phi_{\ell_2}\cdots\phi_{\ell_n} &= \tilde{\gamma}_{k_1k_2\cdots k_n}\,\tilde{\phi}_{k_1}\tilde{\phi}_{k_2}\cdots\tilde{\phi}_{k_n}\,, \\ \tilde{\gamma}_{k_1k_2\cdots k_n} &= \gamma_{\ell_1\ell_2\cdots\ell_n}(\varphi_{k_1})_{\ell_1}(\varphi_{k_2})_{\ell_2}\cdots(\varphi_{k_n})_{\ell_n} \\ &= \frac{1}{N^{n/2}}\sum_{\ell_1\cdots\ell_n}\gamma_{\ell_1\ell_2\cdots\ell_n}\,e^{-i\frac{2\pi}{N}(k_1\ell_1+\cdots+k_n\ell_n)}\,. \end{aligned} \tag{H.54}$$

According to (H.49) the matrix $U_{k\ell} = (\varphi_k)_\ell = \frac{1}{\sqrt{N}}e^{i\frac{2\pi}{N}k\ell}$ is a unitary matrix, and the Fourier transform is a linear, unitary transformation $UU^\dagger = \sum P_k = \mathbf{1}$ with Jacobian det $U = 1$. The form of the path integral (**??**) does not change under $\phi \to \tilde{\phi}_k$ transformation, and from the formal point of view, it does not matter whether we compute in the Fourier space or in the configuration space that we started out with. For example, the trace of $\mathbf{M}$ is the trace in either representation

$$\begin{aligned} \operatorname{tr}\mathbf{M} &= \sum_\ell M_{\ell\ell} = \sum_{kk'}\sum_\ell (P_k\mathbf{M}P_{k'})_{\ell\ell} \\ &= \sum_{kk'}\sum_\ell (\varphi_k)_\ell(\varphi_k^\dagger \cdot \mathbf{M} \cdot \varphi_{k'})(\varphi_{k'}^\dagger)_\ell = \sum_{kk'}\delta_{kk'}\tilde{M}_{kk'} = \operatorname{tr}\tilde{\mathbf{M}}\,. \end{aligned} \tag{H.55}$$

From this it follows that $\operatorname{tr}\mathbf{M}^n = \operatorname{tr}\tilde{\mathbf{M}}^n$, and from the tr ln = ln tr relation that det $\mathbf{M}$ = det $\tilde{\mathbf{M}}$. In fact, any scalar combination of ϕ's, J's and couplings, such as the partition function $Z[J]$, has exactly the same form in the configuration and the Fourier space.

OK, a dizzying quantity of indices. But what's the pay-back?

H.5.2 Lattice Laplacian diagonalized

Now use the eigenvalue equation (H.47) to convert $\mathbf{h}$ matrices into scalars. If $\mathbf{M}$ *commutes* with $\mathbf{h}$, then $(\varphi_k^\dagger \cdot \mathbf{M} \cdot \varphi_{k'}) = \tilde{M}_k\delta_{kk'}$, and the matrix $\mathbf{M}$ acts as a multiplication by the scalar $\tilde{M}_k$ on the k-th subspace. For example, for the 1-dimensional version of the lattice Laplacian (H.37) the projection on the k-th subspace is

$$
\begin{aligned}
(\varphi_k^\dagger \cdot \Delta \cdot \varphi_{k'}) &= \frac{2}{a^2}\left(\frac{1}{2}(\omega^{-k}+\omega^{k})-1\right)(\varphi_k^\dagger \cdot \varphi_{k'}) \\
&= \frac{2}{a^2}\left(\cos\left(\frac{2\pi}{N}k\right)-1\right)\delta_{kk'} \qquad \text{(H.56)}
\end{aligned}
$$

In the k-th subspace the bare propagator (**??**) is simply a number, and, in contrast to the mess generated by (**??**), there is nothing to inverting M^{-1}:

$$
(\varphi_{\mathbf{k}}^\dagger \cdot M \cdot \varphi_{\mathbf{k}'}) = (\tilde{G}_0)_{\mathbf{k}}\delta_{\mathbf{k}\mathbf{k}'} = \frac{1}{\beta}\frac{\delta_{\mathbf{k}\mathbf{k}'}}{m_0'^2 - \frac{2c}{a^2}\sum_{\mu=1}^{d}\left(\cos\left(\frac{2\pi}{N}k_\mu\right)-1\right)}\,, \qquad \text{(H.57)}
$$

where $\mathbf{k} = (k_1, k_2, \cdots, k_\mu)$ is a d-dimensional vector in the N^d-dimensional dual lattice.

Going back to the partition function (31.53) and sticking in the factors of $\mathbf{1}$ into the bilinear part of the interaction, we replace the spatial J_ℓ by its Fourier transform $\tilde{J}_k$, and the spatial propagator $(M)_{\ell\ell'}$ by the diagonalized Fourier transformed $(\tilde{G}_0)_k$

$$
J^T \cdot M \cdot J = \sum_{k,k'}(J^T \cdot \varphi_k)(\varphi_k^\dagger \cdot M \cdot \varphi_{k'})(\varphi_{k'}^\dagger \cdot J) = \sum_k \tilde{J}_k^\dagger (\tilde{G}_0)_k \tilde{J}_k\,. \qquad \text{(H.58)}
$$

What's the price? The interaction term $S_I[\phi]$ (which in (31.53) was local in the configuration space) now has a more challenging k dependence in the Fourier transform version (H.54). For example, the locality of the quartic term leads to the 4-vertex *momentum conservation* in the Fourier space

$$
\begin{aligned}
S_I[\phi] &= \frac{1}{4!}\gamma_{\ell_1\ell_2\ell_3\ell_4}\;\phi_{\ell_1}\phi_{\ell_2}\phi_{\ell_3}\phi_{\ell_4} = -\beta u \sum_{\ell=1}^{N^d}(\phi_\ell)^4 \;\Rightarrow \\
&= -\beta u \frac{1}{N^{3d/2}}\sum_{\{\mathbf{k}_i\}}^{N}\delta_{0,\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4}\,\tilde{\phi}_{\mathbf{k}_1}\tilde{\phi}_{\mathbf{k}_2}\tilde{\phi}_{\mathbf{k}_3}\tilde{\phi}_{\mathbf{k}_4}\,. \qquad \text{(H.59)}
\end{aligned}
$$

H.6 C_{4v} factorization

If an N-disk arrangement has C_N symmetry, and the disk visitation sequence is given by disk labels $\{\epsilon_1\epsilon_2\epsilon_3\ldots\}$, only the relative increments $\rho_i = \epsilon_{i+1} - \epsilon_i \bmod N$ matter. Symmetries under reflections across axes increase the group to C_{Nv} and add relations between symbols: $\{\epsilon_i\}$ and $\{N-\epsilon_i\}$ differ

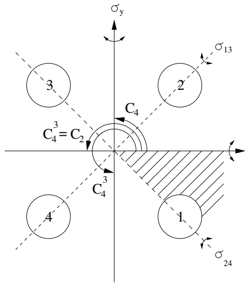


Figure H.1: Symmetries of four disks on a square. A fundamental domain indicated by the shaded wedge.

only by a reflection. As a consequence of this reflection increments become decrements until the next reflection and vice versa. Consider four equal disks placed on the vertices of a square (figure H.1). The symmetry group consists of the identity $\mathbf{e}$, the two reflections σ_x, σ_y across x, y axes, the two diagonal reflections σ_{13}, σ_{24}, and the three rotations C_4, C_2 and C_4^3 by angles $\pi/2$, π and $3\pi/2$. We start by exploiting the C_4 subgroup symmetry in order to replace the absolute labels $\epsilon_i \in \{1,2,3,4\}$ by relative increments $\rho_i \in \{1,2,3\}$. By reflection across diagonals, an increment by 3 is equivalent to an increment by 1 and a reflection; this new symbol will be called $\underline{1}$. Our convention will be to first perform the increment and then to change the orientation due to the reflection. As an example, consider the fundamental domain cycle 112. Taking the disk 1 $\rightarrow$ disk 2 segment as the starting segment, this symbol string is mapped into the disk visitation sequence $1_{+1}2_{+1}3_{+2}1\ldots = \overline{123}$, where the subscript indicates the increments (or decrements) between neighboring symbols; the period of the cycle $\overline{112}$ is thus 3 in both the fundamental domain and the full space. Similarly, the cycle $\overline{\underline{1}12}$ will be mapped into $1_{+1}2_{-1}1_{-2}3_{-1}2_{+1}3_{+2}1 = \overline{121323}$ (note that the fundamental domain symbol $\underline{1}$ corresponds to a flip in orientation after the second and fifth symbols); this time the period in the full space is twice that of the fundamental domain. In particular, the fundamental domain fixed points correspond to the following 4-disk cycles:

4-disk		reduced
12	$\leftrightarrow$	$\underline{1}$
1234	$\leftrightarrow$	1
13	$\leftrightarrow$	2

Conversions for all periodic orbits of reduced symbol period less than 5 are listed in table H.1.

This symbolic dynamics is closely related to the group-theoretic structure of the dynamics: the global 4-disk trajectory can be generated by mapping the fundamental domain trajectories onto the full 4-disk space by the accumulated product of the C_{4v} group elements $g_1 = C$, $g_2 = C^2$, $g_{\underline{1}} = \sigma_{diag}C = \sigma_{axis}$, where C is a rotation by $\pi/2$. In the $\overline{\underline{1}12}$ example worked out above, this yields $g_{\underline{1}12} = g_2 g_1 g_{\underline{1}} = C^2 C \sigma_{axis} = \sigma_{diag}$, listed in the last

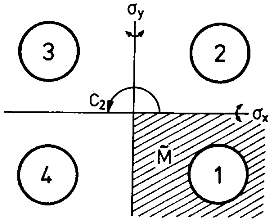


Figure H.2: Symmetries of four disks on a rectangle. A fundamental domain indicated by the shaded wedge.

$\tilde{p}$	p	$\mathbf{h}_{\tilde{p}}$
0	1 2	σ_x
1	1 2 3 4	C_4
2	1 3	C_2, σ_{13}
01	12 14	σ_{24}
02	12 43	σ_y
12	12 41 34 23	C_4^3
001	121 232 343 414	C_4
002	121 343	C_2
011	121 434	σ_y
012	121 323	σ_{13}
021	124 324	σ_{13}
022	124 213	σ_x
112	123	e
122	124 231 342 413	C_4

$\tilde{p}$	p	$\mathbf{h}_{\tilde{p}}$
0001	1212 1414	σ_{24}
0002	1212 4343	σ_y
0011	1212 3434	C_2
0012	1212 4141 3434 2323	C_4^3
0021 (a)	1213 4142 3431 2324	C_4^3
0022	1213	e
0102 (a)	1214 2321 3432 4143	C_4
0111	1214 3234	σ_{13}
0112 (b)	1214 2123	σ_x
0121 (b)	1213 2124	σ_x
0122	1213 1413	σ_{24}
0211	1243 2134	σ_x
0212	1243 1423	σ_{24}
0221	1242 1424	σ_{24}
0222	1242 4313	σ_y
1112	1234 2341 3412 4123	C_4
1122	1231 3413	C_2
1222	1242 4131 3424 2313	C_4^3

Table H.1: C_{4v} correspondence between the ternary fundamental domain prime cycles $\tilde{p}$ and the full 4-disk {1,2,3,4} labeled cycles p, together with the C_{4v} transformation that maps the end point of the $\tilde{p}$ cycle into an irreducible segment of the p cycle. For typographical convenience, the symbol $\underline{1}$ of sect. H.6 has been replaced by 0, so that the ternary alphabet is $\{0,1,2\}$. The degeneracy of the p cycle is $m_p = 8n_{\tilde{p}}/n_p$. Orbit $\overline{2}$ is the sole boundary orbit, invariant both under a rotation by π and a reflection across a diagonal. The two pairs of cycles marked by (a) and (b) are related by time reversal, but cannot be mapped into each other by C_{4v} transformations.

column of table H.1. Our convention is to multiply group elements in the reverse order with respect to the symbol sequence. We need these group elements for our next step, the dynamical zeta function factorizations.

The C_{4v} group has four one-dimensional representations, either symmetric (A_1) or antisymmetric (A_2) under both types of reflections, or symmetric under one and antisymmetric under the other (B_1, B_2), and a degenerate pair of two-dimensional representations E. Substituting the C_{4v} characters

C_{4v}	A_1	A_2	B_1	B_2	E
e	1	1	1	1	2
C_2	1	1	1	1	-2
C_4, C_4^3	1	1	-1	-1	0
σ_{axes}	1	-1	1	-1	0
σ_{diag}	1	-1	-1	1	0

into (**??**) we obtain:

$$
\begin{array}{rclccccc}
h_{\tilde{p}} & & & A_1 & A_2 & B_1 & B_2 & E \\
e: & (1-t_{\tilde{p}})^8 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}})^4 \\
C_2: & (1-t_{\tilde{p}}^2)^4 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}})^4 \\
C_4, C_4^3: & (1-t_{\tilde{p}}^4)^2 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1+t_{\tilde{p}}^2)^2 \\
\sigma_{axes}: & (1-t_{\tilde{p}}^2)^4 & = & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}}^2)^2 \\
\sigma_{diag}: & (1-t_{\tilde{p}}^2)^4 & = & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}^2)^2
\end{array}
$$

The possible irreducible segment group elements $\mathbf{h}_{\tilde{p}}$ are listed in the first column; σ_{axes} denotes a reflection across either the x-axis or the y-axis, and σ_{diag} denotes a reflection across a diagonal (see figure H.1). In addition, degenerate pairs of boundary orbits can run along the symmetry lines in the full space, with the fundamental domain group theory weights $\mathbf{h}_p = (C_2 + \sigma_x)/2$ (axes) and $\mathbf{h}_p = (C_2 + \sigma_{13})/2$ (diagonals) respectively:

$$
\begin{array}{rclc}
 & & & \quad A_1 \qquad A_2 \qquad B_1 \qquad B_2 \qquad E \\
\text{axes:} & (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}})(1-0t_{\tilde{p}})(1-t_{\tilde{p}})(1-0t_{\tilde{p}})(1+t_{\tilde{p}})^2 \\
\text{diagonals:} & (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}})(1-0t_{\tilde{p}})(1-0t_{\tilde{p}})(1-t_{\tilde{p}})(1+t_{\tilde{p}})^2
\end{array}
\qquad (H.60)
$$

(we have assumed that $t_{\tilde{p}}$ does not change sign under reflections across symmetry axes). For the 4-disk arrangement considered here only the diagonal orbits $\overline{13}$, $\overline{24}$ occur; they correspond to the $\overline{2}$ fixed point in the fundamental domain.

The A_1 subspace in C_{4v} cycle expansion is given by

$$
\begin{aligned}
1/\zeta_{A_1} &= (1-t_0)(1-t_1)(1-t_2)(1-t_{01})(1-t_{02})(1-t_{12}) \\
&\quad (1-t_{001})(1-t_{002})(1-t_{011})(1-t_{012})(1-t_{021})(1-t_{022})(1-t_{112})
\end{aligned}
$$

$$
\begin{aligned}
& (1-t_{122})(1-t_{0001})(1-t_{0002})(1-t_{0011})(1-t_{0012})(1-t_{0021})\ldots \\
= \; & 1-t_0-t_1-t_2-(t_{01}-t_0t_1)-(t_{02}-t_0t_2)-(t_{12}-t_1t_2) \\
& -(t_{001}-t_0t_{01})-(t_{002}-t_0t_{02})-(t_{011}-t_1t_{01}) \\
& -(t_{022}-t_2t_{02})-(t_{112}-t_1t_{12})-(t_{122}-t_2t_{12}) \\
& -(t_{012}+t_{021}+t_0t_1t_2-t_0t_{12}-t_1t_{02}-t_2t_{01})\ldots \qquad \text{(H.61)}
\end{aligned}
$$

(for typographical convenience, $\underline{1}$ is replaced by 0 in the remainder of this section). For one-dimensional representations, the characters can be read off the symbol strings: $\chi_{A_2}(\mathbf{h}_{\tilde{\mathbf{p}}}) = (-1)^{n_0}$, $\chi_{B_1}(\mathbf{h}_{\tilde{\mathbf{p}}}) = (-1)^{n_1}$, $\chi_{B_2}(\mathbf{h}_{\tilde{\mathbf{p}}}) = (-1)^{n_0+n_1}$, where n_0 and n_1 are the number of times symbols 0, 1 appear in the $\tilde{p}$ symbol string. For B_2 all t_p with an odd total number of 0's and 1's change sign:

$$
\begin{aligned}
1/\zeta_{B_2} = \; & (1+t_0)(1+t_1)(1-t_2)(1-t_{01})(1+t_{02})(1+t_{12}) \\
& (1+t_{001})(1-t_{002})(1+t_{011})(1-t_{012})(1-t_{021})(1+t_{022})(1-t_{112}) \\
& (1+t_{122})(1-t_{0001})(1+t_{0002})(1-t_{0011})(1+t_{0012})(1+t_{0021})\ldots \\
= \; & 1+t_0+t_1-t_2-(t_{01}-t_0t_1)+(t_{02}-t_0t_2)+(t_{12}-t_1t_2) \\
& +(t_{001}-t_0t_{01})-(t_{002}-t_0t_{02})+(t_{011}-t_1t_{01}) \\
& +(t_{022}-t_2t_{02})-(t_{112}-t_1t_{12})+(t_{122}-t_2t_{12}) \\
& -(t_{012}+t_{021}+t_0t_1t_2-t_0t_{12}-t_1t_{02}-t_2t_{01})\ldots \qquad \text{(H.62)}
\end{aligned}
$$

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The form of the remaining cycle expansions depends crucially on the special role played by the boundary orbits: by (H.60) the orbit t_2 does not contribute to A_2 and B_1,

$$
\begin{aligned}
1/\zeta_{A_2} = \; & (1+t_0)(1-t_1)(1+t_{01})(1+t_{02})(1-t_{12}) \\
& (1-t_{001})(1-t_{002})(1+t_{011})(1+t_{012})(1+t_{021})(1+t_{022})(1-t_{112}) \\
& (1-t_{122})(1+t_{0001})(1+t_{0002})(1-t_{0011})(1-t_{0012})(1-t_{0021})\ldots \\
= \; & 1+t_0-t_1+(t_{01}-t_0t_1)+t_{02}-t_{12} \\
& -(t_{001}-t_0t_{01})-(t_{002}-t_0t_{02})+(t_{011}-t_1t_{01}) \\
& +t_{022}-t_{122}-(t_{112}-t_1t_{12})+(t_{012}+t_{021}-t_0t_{12}-t_1t_{02})\ldots \qquad \text{(H.63)}
\end{aligned}
$$

and

$$
\begin{aligned}
1/\zeta_{B_1} = \; & (1-t_0)(1+t_1)(1+t_{01})(1-t_{02})(1+t_{12}) \\
& (1+t_{001})(1-t_{002})(1-t_{011})(1+t_{012})(1+t_{021})(1-t_{022})(1-t_{112}) \\
& (1+t_{122})(1+t_{0001})(1-t_{0002})(1-t_{0011})(1+t_{0012})(1+t_{0021})\ldots \\
= \; & 1-t_0+t_1+(t_{01}-t_0t_1)-t_{02}+t_{12} \\
& +(t_{001}-t_0t_{01})-(t_{002}-t_0t_{02})-(t_{011}-t_1t_{01}) \\
& -t_{022}+t_{122}-(t_{112}-t_1t_{12})+(t_{012}+t_{021}-t_0t_{12}-t_1t_{02})\ldots \qquad \text{(H.64)}
\end{aligned}
$$

In the above we have assumed that t_2 does not change sign under C_{4v} reflections. For the mixed-symmetry subspace E the curvature expansion

is given by

$$
\begin{aligned}
1/\zeta_E = \; & 1+t_2+(-{t_0}^2+{t_1}^2)+(2t_{002}-t_2{t_0}^2-2t_{112}+t_2{t_1}^2) \\
& +(2t_{0011}-2t_{0022}+2t_2t_{002}-{t_{01}}^2-{t_{02}}^2+2t_{1122}-2t_2t_{112} \\
& +{t_{12}}^2-{t_0}^2{t_1}^2)+(2t_{00002}-2t_{00112}+2t_2t_{0011}-2t_{00121}-2t_{00211} \\
& +2t_{00222}-2t_2t_{0022}+2t_{01012}+2t_{01021}-2t_{01102}-t_2{t_{01}}^2+2t_{02022} \\
& -t_2{t_{02}}^2+2t_{11112}-2t_{11222}+2t_2t_{1122}-2t_{12122}+t_2{t_{12}}^2-t_2{t_0}^2{t_1}^2 \\
& +2t_{002}(-{t_0}^2+{t_1}^2)-2t_{112}(-{t_0}^2+{t_1}^2)) \qquad \text{(H.65)}
\end{aligned}
$$

A quick test of the $\zeta = \zeta_{A_1}\zeta_{A_2}\zeta_{B_1}\zeta_{B_2}\zeta_E^2$ factorization is afforded by the topological polynomial; substituting $t_p = z^{n_p}$ into the expansion yields

$$
1/\zeta_{A_1} = 1-3z\,, \quad 1/\zeta_{A_2} = 1/\zeta_{B_1} = 1\,, \quad 1/\zeta_{B_2} = 1/\zeta_E = 1+z\,,
$$

in agreement with (13.40).

H.7 C_{2v} factorization

An arrangement of four identical disks on the vertices of a rectangle has C_{2v} symmetry (figure H.2b). C_{2v} consists of $\{e, \sigma_x, \sigma_y, C_2\}$, i.e., the reflections across the symmetry axes and a rotation by π.

This system affords a rather easy visualization of the conversion of a 4-disk dynamics into a fundamental domain symbolic dynamics. An orbit leaving the fundamental domain through one of the axis may be folded back by a reflection on that axis; with these symmetry operations $g_0 = \sigma_x$ and $g_1 = \sigma_y$ we associate labels 1 and 0, respectively. Orbits going to the diagonally opposed disk cross the boundaries of the fundamental domain twice; the product of these two reflections is just $C_2 = \sigma_x\sigma_y$, to which we assign the label 2. For example, a ternary string $0\,0\,1\,0\,2\,0\,1\ldots$ is converted into 12143123..., and the associated group-theory weight is given by $\ldots g_1g_0g_2g_0g_1g_0g_0$.

Short ternary cycles and the corresponding 4-disk cycles are listed in table H.2. Note that already at length three there is a pair of cycles ($012 = 143$ and $021 = 142$) related by time reversal, but *not* by any C_{2v} symmetries.

The above is the complete description of the symbolic dynamics for 4 sufficiently separated equal disks placed at corners of a rectangle. However, if the fundamental domain requires further partitioning, the ternary description is insufficient. For example, in the stadium billiard fundamental domain one has to distinguish between bounces off the straight and the curved sections of the billiard wall; in that case five symbols suffice for constructing the covering symbolic dynamics.

$\tilde{p}$	p	**g**
0	1 4	σ_y
1	1 2	σ_x
2	1 3	C_2
01	14 32	C_2
02	14 23	σ_x
12	12 43	σ_y
001	141 232	σ_x
002	141 323	C_2
011	143 412	σ_y
012	143	e
021	142	e
022	142 413	σ_y
112	121 343	C_2
122	124 213	σ_x

$\tilde{p}$	p	**g**
0001	1414 3232	C_2
0002	1414 2323	σ_x
0011	1412	e
0012	1412 4143	σ_y
0021	1413 4142	σ_y
0022	1413	e
0102	1432 4123	σ_y
0111	1434 3212	C_2
0112	1434 2343	σ_x
0121	1431 2342	σ_x
0122	1431 3213	C_2
0211	1421 2312	σ_x
0212	1421 3243	C_2
0221	1424 3242	C_2
0222	1424 2313	σ_x
1112	1212 4343	σ_y
1122	1213	e
1222	1242 4313	σ_y

Table H.2: C_{2v} correspondence between the ternary $\{0,1,2\}$ fundamental domain prime cycles $\tilde{p}$ and the full 4-disk {1,2,3,4} cycles p, together with the C_{2v} transformation that maps the end point of the $\tilde{p}$ cycle into an irreducible segment of the p cycle. The degeneracy of the p cycle is $m_p = 4n_{\tilde{p}}/n_p$. Note that the 012 and 021 cycles are related by time reversal, but cannot be mapped into each other by C_{2v} transformations. The full space orbit listed here is generated from the symmetry reduced code by the rules given in sect. H.7, starting from disk 1.

The group C_{2v} has four one-dimensional representations, distinguished by their behavior under axis reflections. The A_1 representation is symmetric with respect to both reflections; the A_2 representation is antisymmetric with respect to both. The B_1 and B_2 representations are symmetric under one and antisymmetric under the other reflection. The character table is

C_{2v}	A_1	A_2	B_1	B_2
e	1	1	1	1
C_2	1	1	-1	-1
σ_x	1	-1	1	-1
σ_y	1	-1	-1	1

Substituted into the factorized determinant (**??**), the contributions of periodic orbits split as follows

$$
\begin{array}{rlcllll}
g_{\tilde{p}} & & & A_1 & A_2 & B_1 & B_2 \\
e\text{:} & (1-t_{\tilde{p}})^4 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) \\
C_2\text{:} & (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) \\
\sigma_x\text{:} & (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) \\
\sigma_y\text{:} & (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}})
\end{array}
$$

Cycle expansions follow by substituting cycles and their group theory factors from table H.2. For A_1 all characters are $+1$, and the corresponding cycle expansion is given in (H.61). Similarly, the totally antisymmetric subspace

factorization A_2 is given by (H.62), the B_2 factorization of C_{4v}. For B_1 all t_p with an odd total number of 0's and 2's change sign:

$$
\begin{aligned}
1/\zeta_{B_1} = & \; (1+t_0)(1-t_1)(1+t_2)(1+t_{01})(1-t_{02})(1+t_{12}) \\
& (1-t_{001})(1+t_{002})(1+t_{011})(1-t_{012})(1-t_{021})(1+t_{022})(1+t_{112}) \\
& (1-t_{122})(1+t_{0001})(1-t_{0002})(1-t_{0011})(1+t_{0012})(1+t_{0021})\ldots \\
= & \; 1+t_0-t_1+t_2+(t_{01}-t_0t_1)-(t_{02}-t_0t_2)+(t_{12}-t_1t_2) \\
& -(t_{001}-t_0t_{01})+(t_{002}-t_0t_{02})+(t_{011}-t_1t_{01}) \\
& +(t_{022}-t_2t_{02})+(t_{112}-t_1t_{12})-(t_{122}-t_2t_{12}) \\
& -(t_{012}+t_{021}+t_0t_1t_2-t_0t_{12}-t_1t_{02}-t_2t_{01})\ldots
\end{aligned}
\tag{H.66}
$$

For B_2 all t_p with an odd total number of 1's and 2's change sign:

$$
\begin{aligned}
1/\zeta_{B_2} = & \; (1-t_0)(1+t_1)(1+t_2)(1+t_{01})(1+t_{02})(1-t_{12}) \\
& (1+t_{001})(1+t_{002})(1-t_{011})(1-t_{012})(1-t_{021})(1-t_{022})(1+t_{112}) \\
& (1+t_{122})(1+t_{0001})(1+t_{0002})(1-t_{0011})(1-t_{0012})(1-t_{0021})\ldots \\
= & \; 1-t_0+t_1+t_2+(t_{01}-t_0t_1)+(t_{02}-t_0t_2)-(t_{12}-t_1t_2) \\
& +(t_{001}-t_0t_{01})+(t_{002}-t_0t_{02})-(t_{011}-t_1t_{01}) \\
& -(t_{022}-t_2t_{02})+(t_{112}-t_1t_{12})+(t_{122}-t_2t_{12}) \\
& -(t_{012}+t_{021}+t_0t_1t_2-t_0t_{12}-t_1t_{02}-t_2t_{01})\ldots
\end{aligned}
\tag{H.67}
$$

Note that all of the above cycle expansions group long orbits together with their pseudoorbit shadows, so that the shadowing arguments for convergence still apply.

The topological polynomial factorizes as

$$
\frac{1}{\zeta_{A_1}} = 1-3z \quad , \quad \frac{1}{\zeta_{A_2}} = \frac{1}{\zeta_{B_1}} = \frac{1}{\zeta_{B_2}} = 1+z,
$$

consistent with the 4-disk factorization (13.40).

H.8 Hénon map symmetries

We note here a few simple symmetries of the Hénon map (3.15). For $b \neq 0$ the Hénon map is reversible: the backward iteration of (3.16) is given by

$$
x_{n-1} = -\frac{1}{b}(1-ax_n^2-x_{n+1})\,. \tag{H.68}
$$

Hence the time reversal amounts to $b \to 1/b$, $a \to a/b^2$ symmetry in the parameter plane, together with $x \to -x/b$ in the coordinate plane, and there is no need to explore the (a,b) parameter plane outside the strip $b \in \{-1,1\}$. For $b=-1$ the map is orientation and area preserving (see (21.1) below),

$$x_{n-1} = 1 - ax_n^2 - x_{n+1}\,, \tag{H.69}$$

the backward and the forward iteration are the same, and the non–wandering set is symmetric across the $x_{n+1} = x_n$ diagonal. This is one of the simplest models of a Poincaré return map for a Hamiltonian flow. For the orientation reversing $b = 1$ case we have

$$x_{n-1} = 1 - ax_n^2 + x_{n+1}\,, \tag{H.70}$$

and the non–wandering set is symmetric across the $x_{n+1} = -x_n$ diagonal.

Commentary

Remark H.1 Literature This material is covered in any introduction to linear algebra [**?**, **?**, **?**] or group theory [9.15, 9.16]. The exposition given in sects. H.2.1 and H.2.2 is taken from refs. [**?**, **?**, **?**]. Who wrote this down first we do not know, but we like Harter's exposition [**?**, **?**, 4.8] best. Harter's theory of class algebrasoffers a more elegant and systematic way of constructing the maximal set of commuting invariant matrices $\mathbf{M}_i$ than the sketch offered in this section.

Remark H.2 Labeling conventions While there is a variety of labeling conventions [23.15, 9.11] for the reduced C_{4v} dynamics, we prefer the one introduced here because of its close relation to the group-theoretic structure of the dynamics: the global 4-disk trajectory can be generated by mapping the fundamental domain trajectories onto the full 4-disk space by the accumulated product of the C_{4v} group elements.

Remark H.3 C_{2v} symmetry C_{2v} is the symmetry of several systems studied in the literature, such as the stadium billiard [8.9], and the 2-dimensional anisotropic Kepler potential [32.6].