Chapter 13

Counting

> That which is crooked cannot be made straight: and that which is wanting cannot be numbered.
>
> —Ecclestiastes 1.15

We are now in a position to apply the periodic orbit theory to the first and the easiest problem in theory of chaotic systems: cycle counting. This is the simplest illustration of the raison d'etre of periodic orbit theory; we shall develop a duality transformation that relates *local* information - in this case the next admissible symbol in a symbol sequence - to *global* averages, in this case the mean rate of growth of the number of admissible itineraries with increasing itinerary length. We shall transform the topological dynamics of chapter 10 into a multiplicative operation by means of transition matrices/Markov graphs, and show that the nth power of a transition matrix counts all itineraries of length n. The asymptotic growth rate of the number of admissible itineraries is therefore given by the leading eigenvalue of the transition matrix; the leading eigenvalue is in turn given by the leading zero of the characteristic determinant of the transition matrix, which is - in this context - called the *topological zeta function*. For flows with finite Markov graphs this determinant is a finite polynomial which can be read off the Markov graph.

The method goes well beyond the problem at hand, and forms the core of the entire treatise, making tangible a rather abstract notion of "spectral determinants" yet to come.

13.1 How many ways to get there from here?

In the 3-disk system the number of admissible trajectories doubles with every iterate: there are $K_n = 3 \cdot 2^n$ distinct itineraries of length n. If disks are too close and some part of trajectories is pruned, this is only an upper bound and explicit formulas might be hard to discover, but we still might be able to establish a lower exponential bound of the form $K_n \geq Ce^{n\hat{h}}$.

Bounded exponentially by $3e^{n \ln 2} \geq K_n \geq Ce^{n\hat{h}}$, the number of trajectories must grow exponentially as a function of the itinerary length, with rate given by the *topological entropy*:

$$h = \lim_{n\to\infty} \frac{1}{n} \ln K_n \,. \tag{13.1}$$

We shall now relate this quantity to the spectrum of the transition matrix, with the growth rate of the number of topologically distinct trajectories given by the leading eigenvalue of the transition matrix.

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The transition matrix element $T_{ij} \in \{0,1\}$ in (10.2) indicates whether the transition from the starting partition j into partition i in one step is allowed or not, and the (i,j) element of the transition matrix iterated n times

$$(T^n)_{ij} = \sum_{k_1,k_2,\ldots,k_{n-1}} T_{ik_1} T_{k_1k_2} \ldots T_{k_{n-1}j}$$

receives a contribution 1 from every admissible sequence of transitions, so $(T^n)_{ij}$ is the number of admissible n symbol itineraries starting with j and ending with i.

Example 13.1 3-disk itinerary counting.

The $(T^2)_{13} = 1$ element of T^2 for the 3-disk transition matrix (10.5)

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} . \tag{13.2}$$

corresponds to $3 \to 2 \to 1$, the only 2-step path from 3 to 1, while $(T^2)_{33} = 2$ counts the two itineraries 313 and 323.

The total number of admissible itineraries of n symbols is

$$K_n = \sum_{ij} (T^n)_{ij} = (\,1, 1, \ldots, 1\,)\; T^n \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} . \tag{13.3}$$

We can also count the number of prime cycles and pruned periodic points, but in order not to break up the flow of the main argument, we relegate these pretty results to sects. 13.5.2 and 13.7. Recommended reading if you ever have to compute lots of cycles.

The matrix T has non-negative integer entries. A matrix M is said to be *Perron-Frobenius* if some power k of M has strictly positive entries,

$(M^k)_{rs} > 0$. In the case of the transition matrix T this means that every partition eventually reaches all of the partitions, i.e., the partition is dynamically transitive or indecomposable, as assumed in (2.2). The notion of *transitivity* is crucial in ergodic theory: a mapping is transitive if it has a dense orbit. This notion is inherited by the shift operation once we introduce a symbolic dynamics. If that is not the case, state space decomposes into disconnected pieces, each of which can be analyzed separately by a separate indecomposable Markov graph. Hence it suffices to restrict our considerations to transition matrices of Perron-Frobenius type.

A finite $[N \times N]$ matrix T has eigenvalues $T\varphi_\alpha = \lambda_\alpha \varphi_\alpha$ and (right) eigenvectors $\{\varphi_0, \varphi_1, \cdots, \varphi_{M-1}\}$. Expressing the initial vector in (13.3) in this basis (which might be incomplete, $M \leq N$),

$$T^n \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = T^n \sum_{\alpha=0}^{N-1} b_\alpha \varphi_\alpha = \sum_{\alpha=0}^{N-1} b_\alpha \lambda_\alpha^n \varphi_\alpha \,,$$

and contracting with $(1, 1, \ldots, 1)$, we obtain

$$K_n = \sum_{\alpha=0}^{N-1} c_\alpha \lambda_\alpha^n \,.$$

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The constants c_α depend on the choice of initial and final partitions: In this example we are sandwiching T^n between the vector $(1, 1, \ldots, 1)$ and its transpose, but any other pair of vectors would do, as long as they are not orthogonal to the leading eigenvector φ_0. In a experiment the vector $(1, 1, \ldots, 1)$ would be replaced by a description of the initial state,and the right vector would describe the measure time n later.

Perron theorem states that a Perron-Frobenius matrix has a nondegenerate positive real eigenvalue $\lambda_0 > 1$ (with a positive eigenvector) which exceeds the moduli of all other eigenvalues. Therefore as n increases, the sum is dominated by the leading eigenvalue of the transition matrix, $\lambda_0 > |\mathrm{Re}\, \lambda_\alpha|$, $\alpha = 1, 2, \cdots, N-1$, and the topological entropy (13.1) is given by

$$\begin{aligned} h &= \lim_{n\to\infty} \frac{1}{n} \ln c_0 \lambda_0^n \left[1 + \frac{c_1}{c_0} \left(\frac{\lambda_1}{\lambda_0} \right)^n + \cdots \right] \\ &= \ln \lambda_0 + \lim_{n\to\infty} \left[\frac{\ln c_0}{n} + \frac{1}{n} \frac{c_1}{c_0} \left(\frac{\lambda_1}{\lambda_0} \right)^n + \cdots \right] \\ &= \ln \lambda_0 \,. \end{aligned} \tag{13.4}$$

What have we learned? The transition matrix T is a one-step *short time* operator, advancing the trajectory from a partition to the next admissible partition. Its eigenvalues describe the rate of growth of the total number

of trajectories at the *asymptotic times*. Instead of painstakingly counting $K_1, K_2, K_3, \ldots$ and estimating (13.1) from a slope of a log-linear plot, we have the *exact* topological entropy if we can compute the leading eigenvalue of the transition matrix T. This is reminiscent of the way the free energy is computed from transfer matrix for one-dimensional lattice models with finite range interactions. Historically, it is analogy with statistical mechanics that led to introduction of evolution operator methods into the theory of chaotic systems, theory that will be developed further in chapter 21.

13.2 Topological trace formula

There are two standard ways of getting at eigenvalues of a matrix - by evaluating the trace $\mathrm{tr}\, T^n = \sum \lambda_\alpha^n$, or by evaluating the determinant $\det(1 - zT)$. We start by evaluating the trace of transition matrices.

Consider an M-step memory transition matrix, like the 1-step memory example (10.13). The trace of the transition matrix counts the number of partitions that map into themselves. In the binary case the trace picks up only two contributions on the diagonal, $T_{0\cdots0,0\cdots0} + T_{1\cdots1,1\cdots1}$, no matter how much memory we assume. We can even take infinite memory $M \to \infty$, in which case the contributing partitions are shrunk to the fixed points, $\mathrm{tr}\, T = T_{\overline{0},\overline{0}} + T_{\overline{1},\overline{1}}$.

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More generally, each closed walk through n concatenated entries of T contributes to $\mathrm{tr}\, T^n$ a product of the matrix entries along the walk. Each step in such a walk shifts the symbolic string by one symbol; the trace ensures that the walk closes on a periodic string c. Define t_c to be the *local trace*, the product of matrix elements along a cycle c, each term being multiplied by a book keeping variable z. $z^n \mathrm{tr}\, T^n$ is then the sum of t_c for all cycles of length n. For example, for an $[8{\times}8]$ transition matrix $T_{s_1s_2s_3,s_0s_1s_2}$ version of (10.13), or any refined partition $[2^n \times 2^n]$ transition matrix, n arbitrarily large, the periodic point $\overline{100}$ contributes $t_{100} = z^3 T_{\overline{100},\overline{010}} T_{\overline{010},\overline{001}} T_{\overline{001},\overline{100}}$ to $z^3 \mathrm{tr}\, T^3$. This product is manifestly cyclically symmetric, $t_{100} = t_{010} = t_{001}$, and so a prime cycle p of length n_p contributes n_p times, once for each periodic point along its orbit. For the binary labeled non–wandering set the first few traces are given by (consult tables 10.1 and 13.1)

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$$\begin{aligned} z\, \mathrm{tr}\, T &= t_0 + t_1, \\ z^2 \mathrm{tr}\, T^2 &= t_0^2 + t_1^2 + 2t_{10}, \\ z^3 \mathrm{tr}\, T^3 &= t_0^3 + t_1^3 + 3t_{100} + 3t_{101}, \\ z^4 \mathrm{tr}\, T^4 &= t_0^4 + t_1^4 + 2t_{10}^2 + 4t_{1000} + 4t_{1001} + 4t_{1011}. \end{aligned} \tag{13.5}$$

For complete binary symbolic dynamics $t_p = z^{n_p}$ for every binary prime cycle p; if there is pruning $t_p = z^{n_p}$ if p is admissible cycle and $t_p = 0$ otherwise. Hence $\mathrm{tr}\, T^n$ counts the number of *admissible periodic points* of

n	N_n	# of prime cycles of length n_p									
		1	2	3	4	5	6	7	8	9	10
1	2	2									
2	4	2	1								
3	8	2		2							
4	16	2	1		3						
5	32	2				6					
6	64	2	1	2			9				
7	128	2						18			
8	256	2	1		3				30		
9	512	2		2						56	
10	1024	2	1			6					99

Table 13.1: The total numbers of periodic points N_n of period n for binary symbolic dynamics. The numbers of prime cycles contributing illustrates the preponderance of long prime cycles of length n over the repeats of shorter cycles of lengths n_p, $n = rn_p$. Further listings of binary prime cycles are given in tables 10.1 and 13.2. (L. Rondoni)

period n. In general, the nth order trace (13.5) picks up contributions from all repeats of prime cycles, with each cycle contributing n_p periodic points, so the total number of periodic points of period n is given by

$$z^n N_n = z^n \mathrm{tr}\, T^n = \sum_{n_p|n} n_p t_p^{n/n_p} = \sum_p n_p \sum_{r=1}^{\infty} \delta_{n,n_p r} t_p^r \,. \tag{13.6}$$

Here $m|n$ means that m is a divisor of n, and (taking $z = 1$) $t_p = 1$ if the cycle is admissible, and $t_p = 0$ otherwise.

In order to get rid of the awkward divisibility constraint $n = n_p r$ in the above sum, we introduce the generating function for numbers of periodic points

$$\sum_{n=1}^{\infty} z^n N_n = \mathrm{tr}\, \frac{zT}{1 - zT} \,. \tag{13.7}$$

Substituting (13.6) into the left hand side, and replacing the right hand side by the eigenvalue sum $\mathrm{tr}\, T^n = \sum \lambda_\alpha^n$, we obtain our first example of a trace formula, the *topological trace formula*

$$\sum_{\alpha=0} \frac{z\lambda_\alpha}{1 - z\lambda_\alpha} = \sum_p \frac{n_p t_p}{1 - t_p} \,. \tag{13.8}$$

A trace formula relates the spectrum of eigenvalues of an operator - in this case the transition matrix - to the spectrum of periodic orbits of the dynamical system. The z^n sum in (13.7) is a discrete version of the Laplace transform (see chapter 16), and the resolvent on the left hand side is the antecedent of the more sophisticated trace formulas (16.10), (16.23) and (32.3). We shall now use this result to compute the spectral determinant of the transition matrix.

13.3 Determinant of a graph

Our next task is to determine the zeros of the *spectral determinant* of an $[M \times M]$ transition matrix

$$\det(1 - zT) = \prod_{\alpha=0}^{M-1} (1 - z\lambda_\alpha) \,. \tag{13.9}$$

We could now proceed to diagonalize T on a computer, and get this over with. It pays, however, to dissect $\det(1-zT)$ with some care; understanding this computation in detail will be the key to understanding the cycle expansion computations of chapter 18 for arbitrary dynamical averages. For T a finite matrix, (13.9) is just the characteristic equation for T. However, we shall be able to compute this object even when the dimension of T and other such operators goes to ∞, and for that reason we prefer to refer to (13.9) loosely as the "spectral determinant".

There are various definitions of the determinant of a matrix; they mostly reduce to the statement that the determinant is a certain sum over all possible permutation cycles composed of the traces $\mathrm{tr}\, T^k$, in the spirit of the determinant–trace relation (1.15):

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$$\begin{aligned} \det(1 - zT) &= \exp(\mathrm{tr}\, \ln(1 - zT)) = \exp\left(-\sum_{n=1} \frac{z^n}{n} \mathrm{tr}\, T^n \right) \\ &= 1 - z\, \mathrm{tr}\, T - \frac{z^2}{2} \left((\mathrm{tr}\ T)^2 - \mathrm{tr}\,(T^2) \right) - \dots \end{aligned} \tag{13.10}$$

This is sometimes called a cumulant expansion. Formally, the right hand is an infinite sum over powers of z^n. If T is an $[M \times M]$ finite matrix, then the characteristic polynomial is at most of order M. In that case the coefficients of z^n, $n > M$ must vanish *exactly.*

We now proceed to relate the determinant in (13.10) to the corresponding Markov graph of chapter 10: to this end we start by the usual algebra textbook expression for a determinant as the sum of products of all permutations

$$\det(1-zT) = \sum_{\{\pi\}} (-1)^\pi (1-zT)_{1,\pi_1} (1-zT)_{2,\pi_2} \cdots (1-zT)_{M,\pi_M} \tag{13.11}$$

where T is a $[M \times M]$ matrix, $\{\pi\}$ denotes the set of permutations of M symbols, π_k is what k is permuted into by the permutation π, and $(-1)^\pi = \pm 1$ is the parity of permutation π. The right hand side of (13.11) yields a polynomial of order M in z: a contribution of order n in z picks up $M - n$ unit factors along the diagonal, the remaining matrix elements yielding

$$(-z)^n (-1)^{\tilde{\pi}} T_{\eta_1, \tilde{\pi}_{\eta_1}} \cdots T_{\eta_n, \tilde{\pi}_{\eta_n}} \tag{13.12}$$

where $\tilde{\pi}$ is the permutation of the subset of n distinct symbols $\eta_1 \dots \eta_n$ indexing T matrix elements. As in (13.5), we refer to any combination $t_c = T_{\eta_1\eta_2}T_{\eta_2\eta_3}\cdots T_{\eta_k\eta_1}$, for a given itinerary $c = \eta_1\eta_2\cdots,\eta_k$, as the *local trace* associated with a closed loop c on the Markov graph. Each term of form (13.12) may be factored in terms of local traces $t_{c_1}t_{c_2}\cdots t_{c_k}$, that is loops on the Markov graph. These loops are non-intersecting, as each node may only be reached by *one* link, and they are indeed loops, as if a node is reached by a link, it has to be the starting point of another *single* link, as each η_j must appear exactly *once* as a row and column index.

So the general structure is clear, a little more thinking is only required to get the sign of a generic contribution. We consider only the case of loops of length 1 and 2, and leave to the reader the task of generalizing the result by induction. Consider first a term in which only loops of unit length appear on (13.12), that is, only the diagonal elements of T are picked up. We have $k = n$ loops and an even permutation $\tilde{\pi}$ so the sign is given by $(-1)^k$, k being the number of loops. Now take the case in which we have i single loops and j loops of length $n = 2j+i$. The parity of the permutation gives $(-1)^j$ and the first factor in (13.12) gives $(-1)^n = (-1)^{2j+i}$. So once again these terms combine into $(-1)^k$, where $k = i + j$ is the number of loops. We may summarize our findings as follows:

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The characteristic polynomial of a transition matrix/Markov graph is given by the sum of all possible partitions π of the graph into products of non-intersecting loops, with each loop trace t_p carrying a minus sign:

$$\det(1 - zT) = \sum_{k=0}^{f} {\sum_\pi}' (-1)^k t_{p_1}\cdots t_{p_k} \qquad (13.13)$$

Any self-intersecting loop is *shadowed* by a product of two loops that share the intersection point. As both the long loop t_{ab} and its shadow $t_a t_b$ in the case at hand carry the same weight $z^{n_a+n_b}$, the cancellation is exact, and the loop expansion (13.13) is finite, with f the maximal number of non-intersecting loops.

We refer to the set of all non-self-intersecting loops $\{t_{p_1}, t_{p_2}, \cdots t_{p_f}\}$ as the *fundamental cycles*. This is not a very good definition, as the Markov graphs are not unique – the most we know is that for a given finite-grammar language, there exist Markov graph(s) with the minimal number of loops. Regardless of how cleverly a Markov graph is constructed, it is always true that for any finite Markov graph the number of fundamental cycles f is finite. If you know a better way to define the "fundamental cycles", let us know.

fast track:
sect. 13.4, p. 243

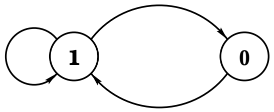

Figure 13.1: The golden mean pruning rule Markov graph, see also figure 10.13.

13.3.1 Topological polynomials: learning by examples

The above definition of the determinant in terms of traces is most easily grasped by working through a few examples. The complete binary dynamics Markov graph of figure 10.11 (b) is a little bit too simple, but let us start humbly.

Example 13.2 Topological polynomial for complete binary dynamics: *There are only two non-intersecting loops, yielding*

$$\det(1 - zT) = 1 - t_0 - t_1 = 1 - 2z\,. \qquad (13.14)$$

The leading (and only) zero of this characteristic polynomial yields the topological entropy $e^h = 2$. As we know that there are $K_n = 2^n$ binary strings of length N, we are not surprised by this result.

Similarly, for complete symbolic dynamics of N symbols the Markov graph has one node and N links, yielding

$$\det(1 - zT) = 1 - Nz\,, \qquad (13.15)$$

whence the topological entropy $h = \ln N$.

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Example 13.3 Golden mean pruning: *A more interesting example is the "golden mean" pruning of figure 13.1. There is only one grammar rule, that a repeat of symbol 0 is forbidden. The non-intersecting loops are of length 1 and 2, so the topological polynomial is given by*

$$\det(1 - zT) = 1 - t_1 - t_{01} = 1 - z - z^2\,. \qquad (13.16)$$

The leading root of this polynomial is the golden mean, so the entropy (13.4) is the logarithm of the golden mean, $h = \ln\frac{1+\sqrt{5}}{2}$.

Example 13.4 Nontrivial pruning: *The non-self-intersecting loops of the Markov graph of figure 13.3 (d) are indicated in figure 13.3 (e). The determinant can be written down by inspection, as the sum of all possible partitions of the graph into products of non-intersecting loops, with each loop carrying a minus sign:*

$$\begin{aligned}\det(1 - zT) &= 1 - t_0 - t_{0011} - t_{0001} - t_{00011} \\ &\quad + t_0 t_{0011} + t_{0011}t_{0001}\,. \end{aligned} \qquad (13.17)$$

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With $t_p = z^{n_p}$, where n_p is the length of the p-cycle, the smallest root of

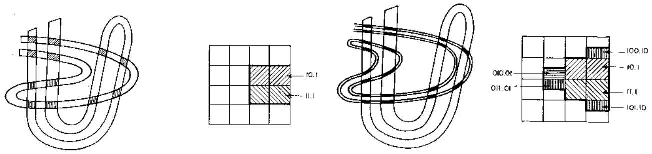

Figure 13.2: (a) An incomplete Smale horseshoe: the inner forward fold does not intersect the two rightmost backward folds. (b) The primary pruned region in the symbol square and the corresponding forbidden binary blocks. (c) An incomplete Smale horseshoe which illustrates (d) the monotonicity of the pruning front: the thick line which delineates the left border of the primary pruned region is monotone on each half of the symbol square. The backward folding in figures (a) and (c) is only schematic - in invertible mappings there are further missing intersections, all obtained by the forward and backward iterations of the primary pruned region.

$$0 = 1 - z - 2z^4 + z^8 \tag{13.18}$$

yields the topological entropy $h = -\ln z$, $z = 0.658779\ldots$, $h = 0.417367\ldots$, *significantly smaller than the entropy of the covering symbolic dynamics, the complete binary shift* $h = \ln 2 \; = \; 0.693\ldots$

in depth:
sect. T.1, p. 939

13.4 Topological zeta function

What happens if there is no finite-memory transition matrix, if the Markov graph is infinite? If we are never sure that looking further into future will reveal no further forbidden blocks? There is still a way to define the determinant, and this idea is central to the whole treatise: the determinant is then defined by its *cumulant* expansion (13.10)

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$$\det(1 - zT) = 1 - \sum_{n=1}^{\infty} \hat{c}_n z^n \,. \tag{13.19}$$

For finite dimensional matrices the expansion is a finite polynomial, and (13.19) is an identity; however, for infinite dimensional operators the cumulant expansion coefficients $\hat{c}_n$ *define* the determinant.

Let us now evaluate the determinant in terms of traces for an arbitrary transition matrix. In order to obtain an expression for the spectral determinant (13.9) in terms of cycles, substitute (13.6) into (13.19) and sum over the repeats of prime cycles using $\ln(1-x) = \sum_r x^r/r$,

$$\det(1 - zT) = \exp\left(-\sum_p \sum_{r=1}^{\infty} \frac{t_p^r}{r}\right) = \prod_p (1 - t_p) \,, \tag{13.20}$$

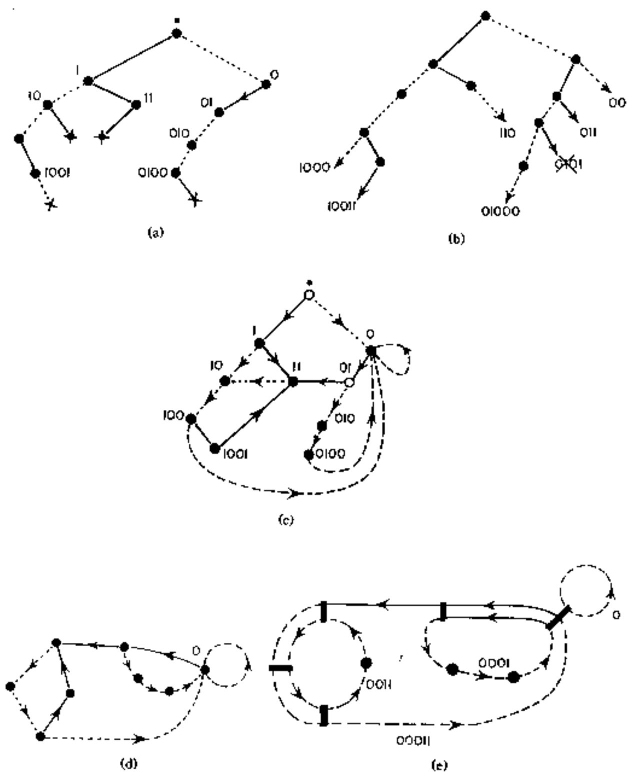

Figure 13.3: Conversion of the pruning front of figure 13.2 (d) into a finite Markov graph. (a) Starting with the start node ".", delineate all pruning blocks on the binary tree. A solid line stands for "1" and a dashed line for "0". Ends of forbidden strings are marked with $\times$. Label all internal nodes by reading the bits connecting ".", the base of the tree, to the node. (b) Indicate all admissible starting blocks by arrows. (c) Drop recursively the leading bits in the admissible blocks; if the truncated string corresponds to an internal node in (a), connect them. (d) Delete the transient, non-circulating nodes; all admissible sequences are generated as walks on this finite Markov graph. (e) Identify all distinct loops and construct the determinant (13.17).

where for the topological entropy the weight assigned to a prime cycle p of length n_p is $t_p = z^{n_p}$ if the cycle is admissible, or $t_p = 0$ if it is pruned. This determinant is called the *topological* or the *Artin-Mazur* zeta function, conventionally denoted by

$$1/\zeta_{\text{top}} = \prod_p (1 - z^{n_p}) = 1 - \sum_{n=1} \hat{c}_n z^n \, . \tag{13.21}$$

Counting cycles amounts to giving each admissible prime cycle p weight $t_p = z^{n_p}$ and expanding the Euler product (13.21) as a power series in z. As the precise expression for coefficients $\hat{c}_n$ in terms of local traces t_p is more general than the current application to counting, we shall postpone its derivation to chapter 18.

The topological entropy h can now be determined from the leading zero $z = e^{-h}$ of the topological zeta function. For a finite $[M \times M]$ transition matrix, the number of terms in the characteristic equation (13.13) is finite, and we refer to this expansion as the *topological polynomial* of order $\leq M$. The power of defining a determinant by the cumulant expansion is that it works even when the partition is infinite, $M \to \infty$; an example is given in sect. 13.6, and many more later on.

fast track:
sect. 13.6, p. 250

13.4.1 Topological zeta function for flows

We now apply the method that we shall use in deriving (16.23) to the problem of deriving the topological zeta functions for flows. The time-weighted density of prime cycles of period t is

$$\Gamma(t) = \sum_p \sum_{r=1} T_p \, \delta(t - rT_p) \, . \tag{13.22}$$

As in (16.22), a Laplace transform smooths the sum over Dirac delta spikes and yields the *topological trace formula*

$$\sum_p \sum_{r=1} T_p \int_{0_+}^{\infty} dt \, e^{-st} \, \delta(t - rT_p) = \sum_p T_p \sum_{r=1}^{\infty} e^{-sT_p r} \tag{13.23}$$

and the *topological zeta function* for flows:

$$1/\zeta_{\text{top}}(s) = \prod_p \left(1 - e^{-sT_p}\right) \, , \tag{13.24}$$

related to the trace formula by

$$\sum_p T_p \sum_{r=1}^{\infty} e^{-sT_p r} = -\frac{\partial}{\partial s} \ln 1/\zeta_{\text{top}}(s) \, .$$

This is the continuous time version of the discrete time topological zeta function (13.21) for maps; its leading zero $s = -h$ yields the topological entropy for a flow.

13.5 Counting cycles

In what follows we shall occasionally need to compute all cycles up to topological length n, so it is handy to know their exact number.

13.5.1 Counting periodic points

N_n, the number of periodic points of period n can be computed from (13.19) and (13.7) as a logarithmic derivative of the topological zeta function

$$\begin{aligned} \sum_{n=1} N_n z^n &= \operatorname{tr}\left(-z\frac{d}{dz}\ln(1 - zT)\right) = -z\frac{d}{dz}\ln\det\,(1 - zT) \\ &= \frac{-z\frac{d}{dz}1/\zeta_{\text{top}}}{1/\zeta_{\text{top}}} \, . \end{aligned} \tag{13.25}$$

We see that the trace formula (13.8) diverges at $z \to e^{-h}$, as the denominator has a simple zero there.

Example 13.5 Complete N-ary dynamics: *As a check of formula (13.19) in the finite grammar context, consider the complete N-ary dynamics (10.3) for which the number of periodic points of period n is simply $\operatorname{tr} T_c^n = N^n$. Substituting*

$$\sum_{n=1}^{\infty} \frac{z^n}{n} \operatorname{tr} T_c^n = \sum_{n=1}^{\infty} \frac{(zN)^n}{n} = \ln(1 - zN) \, ,$$

into (13.19) we verify (13.15). The logarithmic derivative formula (13.25) in this case does not buy us much either, we recover

$$\sum_{n=1} N_n z^n = \frac{Nz}{1 - Nz} \, .$$

Example 13.6 Nontrivial pruned dynamics: *Consider the pruning of figure 13.3 (e). Substituting (13.18) we obtain*

$$\sum_{n=1} N_n z^n = \frac{z + 8z^4 - 8z^8}{1 - z - 2z^4 + z^8} \,. \tag{13.26}$$

Now the topological zeta function is not merely a tool for extracting the asymptotic growth of N_n; it actually yields the exact and not entirely trivial recursion relation for the numbers of periodic points: $N_1 = N_2 = N_3 = 1$, $N_n = 2n + 1$ for $n = 4, 5, 6, 7, 8$, and $N_n = N_{n-1} + 2N_{n-4} - N_{n-8}$ for $n > 8$.

13.5.2 Counting prime cycles

Having calculated the number of periodic points, our next objective is to evaluate the number of *prime* cycles M_n for a dynamical system whose symbolic dynamics is built from N symbols. The problem of finding M_n is classical in combinatorics (counting necklaces made out of n beads out of N different kinds) and is easily solved. There are N^n possible distinct strings of length n composed of N letters. These N^n strings include all M_d prime d-cycles whose period d equals or divides n. A prime cycle is a non-repeating symbol string: for example, $p = \overline{011} = \overline{101} = \overline{110} = \ldots 011011 \ldots$ is prime, but $\overline{0101} = 010101 \ldots = \overline{01}$ is not. A prime d-cycle contributes d strings to the sum of all possible strings, one for each cyclic permutation. The total number of possible periodic symbol sequences of length n is therefore related to the number of prime cycles by

$$N_n = \sum_{d|n} d M_d \,, \tag{13.27}$$

where N_n equals $\mathrm{tr}\, T^n$. The number of prime cycles can be computed recursively

$$M_n = \frac{1}{n} \left(N_n - \sum_{d|n}^{d<n} d M_d \right) ,$$

or by the *Möbius inversion formula*

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$$M_n = n^{-1} \sum_{d|n} \mu\left(\frac{n}{d}\right) N_d \,. \tag{13.28}$$

where the Möbius function $\mu(1) = 1$, $\mu(n) = 0$ if n has a squared factor, and $\mu(p_1 p_2 \ldots p_k) = (-1)^k$ if all prime factors are different.

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We list the number of prime cycles up to length 10 for 2-, 3- and 4-letter complete symbolic dynamics in table 13.2. The number of *prime* cycles follows by Möbius inversion (13.28).

n	$M_n(N)$	$M_n(2)$	$M_n(3)$	$M_n(4)$
1	N	2	3	4
2	$N(N-1)/2$	1	3	6
3	$N(N^2-1)/3$	2	8	20
4	$N^2(N^2-1)/4$	3	18	60
5	$(N^5-N)/5$	6	48	204
6	$(N^6-N^3-N^2+N)/6$	9	116	670
7	$(N^7-N)/7$	18	312	2340
8	$N^4(N^4-1)/8$	30	810	8160
9	$N^3(N^6-1)/9$	56	2184	29120
10	$(N^{10}-N^5-N^2+N)/10$	99	5880	104754

Table 13.2: Number of prime cycles for various alphabets and grammars up to length 10. The first column gives the cycle length, the second the formula (13.28) for the number of prime cycles for complete N-symbol dynamics, columns three through five give the numbers for $N = 2, 3$ and 4.

Example 13.7 Counting N-disk periodic points: *A simple example of pruning is the exclusion of "self-bounces" in the N-disk game of pinball. The number of points that are mapped back onto themselves after n iterations is given by $N_n = \mathrm{tr}\, T^n$. The pruning of self-bounces eliminates the diagonal entries, $T_{N-disk} = T_c - \mathbf{1}$, so the number of the N-disk periodic points is*

$$N_n = \mathrm{tr}\, T^n_{N-disk} = (N-1)^n + (-1)^n (N-1) \tag{13.29}$$

(here T_c is the complete symbolic dynamics transition matrix (10.3)). For the N-disk pruned case (13.29) Möbius inversion (13.28) yields

$$\begin{aligned} M_n^{N-disk} &= \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) (N-1)^d + \frac{N-1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) (-1)^d \\ &= M_n^{(N-1)} \quad \text{for} \quad n > 2 \,. \end{aligned} \tag{13.30}$$

There are no fixed points, $M_1^{N-disk} = 0$. The number of periodic points of period 2 is $N^2 - N$, hence there are $M_2^{N-disk} = N(N-1)/2$ prime cycles of length 2; for lengths $n > 2$, the number of prime cycles is the same as for the complete $(N-1)$-ary dynamics of table 13.2.

Example 13.8 Pruning individual cycles: *Consider the 3-disk game of pinball. The prohibition of repeating a symbol affects counting only for the fixed points and the 2-cycles. Everything else is the same as counting for a complete binary dynamics (eq (13.30)). To obtain the topological zeta function, just divide out the binary 1- and 2-cycles $(1 - zt_0)(1 - zt_1)(1 - z^2 t_{01})$ and multiply with the correct 3-disk 2-cycles $(1 - z^2 t_{12})(1 - z^2 t_{13})(1 - z^2 t_{23})$:*

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$$\begin{aligned} 1/\zeta_{3-disk} &= (1 - 2z) \frac{(1 - z^2)^3}{(1 - z)^2 (1 - z^2)} \\ &= (1 - 2z)(1 + z)^2 = 1 - 3z^2 - 2z^3 \,. \end{aligned} \tag{13.31}$$

The factorization reflects the underlying 3-disk symmetry; we shall rederive it in (??). As we shall see in chapter ??, symmetries lead to factorizations of topological polynomials and topological zeta functions.

n	M_n	N_n	S_n	$m_p \cdot \hat{p}$
1	0	0	0	
2	3	6=3·2	1	3·12
3	2	6=2·3	1	2·123
4	3	18=3·2+3·4	1	3·1213
5	6	30=6·5	1	6·12123
6	9	66=3·2+2·3+9·6	2	6·121213 + 3·121323
7	18	126=18·7	3	6·1212123 + 6·1212313 + 6·1213123
8	30	258=3·2+3·4+30·8	6	6·12121213 + 3·12121313 + 6·12121323
				+ 6·12123123 + 6·12123213 + 3·12132123
9	56	510=2·3+56·9	10	6·121212123 + 6·(121212313 + 121212323)
				+ 6·(121213123 + 121213213) + 6·121231323
				+ 6·(121231213 + 121232123) + 2·121232313
				+ 6·121321323
10	99	1022	18	

Table 13.3: List of the 3-disk prime cycles up to length 10. Here n is the cycle length, M_n the number of prime cycles, N_n the number of periodic points and S_n the number of distinct prime cycles under the C_{3v} symmetry (see chapter **??** for further details). Column 3 also indicates the splitting of N_n into contributions from orbits of lengths that divide n. The prefactors in the fifth column indicate the degeneracy m_p of the cycle; for example, 3·12 stands for the three prime cycles $\overline{12}$, $\overline{13}$ and $\overline{23}$ related by $2\pi/3$ rotations. Among symmetry related cycles, a representative $\hat{p}$ which is lexically lowest was chosen. The cycles of length 9 grouped by parenthesis are related by time reversal symmetry, but not by any other C_{3v} transformation.

n	M_n	N_n	S_n	$m_p \cdot \hat{p}$
1	0	0	0	
2	6	12=6·2	2	4·12 + 2·13
3	8	24=8·3	1	8·123
4	18	84=6·2+18·4	4	8·1213 + 4·1214 + 2·1234 + 4·1243
5	48	240=48·5	6	8·(12123 + 12124) + 8·12313
				+ 8·(12134 + 12143) + 8·12413
6	116	732=6·2+8·3+116·6	17	8·121213 + 8·121214 + 8·121234
				+ 8·121243 + 8·121313 + 8·121314
				+ 4·121323 + 8·(121324 + 121423)
				+ 4·121343 + 8·121424 + 4·121434
				+ 8·123124 + 8·123134 + 4·123143
				+ 4·124213 + 8·124243
7	312	2184	39	
8	810	6564	108	

Table 13.4: List of the 4-disk prime cycles up to length 8. The meaning of the symbols is the same as in table 13.3. Orbits related by time reversal symmetry (but no other symmetry) already appear at cycle length 5. List of the cycles of length 7 and 8 has been omitted.

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Example 13.9 Alphabet $\{a, cb^k;\ \overline{b}\}$**:** *(continuation of exercise 13.16) In the cycle counting case, the dynamics in terms of* $a \to z$, $cb^k \to \frac{z}{1-z}$ *is a complete binary dynamics with the explicit fixed point factor* $(1-t_b) = (1-z)$*:*

$$1/\zeta_{top} = (1-z)\left(1 - z - \frac{z}{1-z}\right) = 1 - 3z + z^2 .$$

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13.6 Topological zeta function for an infinite partition

(K.T. Hansen and P. Cvitanović)

Now consider an example of a dynamical system which (as far as we know - there is no proof) has an infinite partition, or an infinity of longer and longer pruning rules. Take the 1-d quadratic map

$$f(x) = Ax(1-x)$$

with $A = 3.8$. It is easy to check numerically that the itinerary or the "kneading sequence" (see figure **??**) of the critical point $x = 1/2$ is

$$K = 1011011110110111101011110111110\ldots$$

where the symbolic dynamics is defined by the partition of figure 10.6. How this kneading sequence is converted into a series of pruning rules is a dark art, relegated to appendix D.1 For the moment it suffices to state the result, to give you a feeling for what a "typical" infinite partition topological zeta function looks like. Approximating the dynamics by a Markov graph corresponding to a repeller of the period 29 attractive cycle close to the $A = 3.8$ strange attractor (or, much easier, following the algorithm of appendix D.1) yields a Markov graph with 29 nodes and the characteristic polynomial

$$\begin{aligned} 1/\zeta^{(29)}_{\mathrm{top}} &= 1 - z^1 - z^2 + z^3 - z^4 - z^5 + z^6 - z^7 + z^8 - z^9 - z^{10} \\ &\quad + z^{11} - z^{12} - z^{13} + z^{14} - z^{15} + z^{16} - z^{17} - z^{18} + z^{19} + z^{20} \\ &\quad - z^{21} + z^{22} - z^{23} + z^{24} + z^{25} - z^{26} + z^{27} - z^{28} . \end{aligned} \tag{13.32}$$

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The smallest real root of this approximate topological zeta function is

$$z = 0.62616120\ldots \tag{13.33}$$

Constructing finite Markov graphs of increasing length corresponding to $A \to 3.8$ we find polynomials with better and better estimates for the

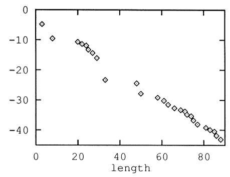

Figure 13.4: The logarithm of the difference between the leading zero of the finite polynomial approximations to topological zeta function and our best estimate, as a function of the length for the quadratic map $A = 3.8$.

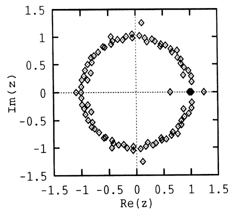

Figure 13.5: The 90 zeroes of the characteristic polynomial for the quadratic map $A = 3.8$ approximated by symbolic strings up to length 90. (from ref. [1.7])

topological entropy. For the closest stable period 90 orbit we obtain our best estimate of the topological entropy of the repeller:

$$h = -\ln 0.62616130424685\ldots = 0.46814726655867\ldots. \tag{13.34}$$

Figure 13.4 illustrates the convergence of the truncation approximations to the topological zeta function as a plot of the logarithm of the difference between the zero of a polynomial and our best estimate (13.34), plotted as a function of the length of the stable periodic orbit. The error of the estimate (13.33) is expected to be of order $z^{29} \approx e^{-14}$ because going from length 28 to a longer truncation yields typically combinations of loops with 29 and more nodes giving terms $\pm z^{29}$ and of higher order in the polynomial. Hence the convergence is exponential, with exponent of $-0.47 = -h$, the topological entropy itself. In figure 13.5 we plot the zeroes of the polynomial approximation to the topological zeta function obtained by accounting for all forbidden strings of length 90 or less. The leading zero giving the topological entropy is the point closest to the origin. Most of the other zeroes are close to the unit circle; we conclude that for infinite Markov partitions the topological zeta function has a unit circle as the radius of convergence. The convergence is controlled by the ratio of the leading to the next-to-leading eigenvalues, which is in this case indeed $\lambda_1/\lambda_0 = 1/e^h = e^{-h}$.

13.7 Shadowing

The topological zeta function is a pretty function, but the infinite product (13.20) should make you pause. For finite transfer matrices the left hand side is a determinant of a finite matrix, therefore a finite polynomial; so

why is the right hand side an infinite product over the infinitely many prime periodic orbits of all periods?

The way in which this infinite product rearranges itself into a finite polynomial is instructive, and crucial for all that follows. You can already take a peek at the full cycle expansion (18.7) of chapter 18; all cycles beyond the fundamental t_0 and t_1 appear in the shadowing combinations such as

$$t_{s_1 s_2 \cdots s_n} - t_{s_1 s_2 \cdots s_m} t_{s_{m+1} \cdots s_n}\,.$$

For subshifts of finite type such shadowing combinations cancel *exactly*, if we are counting cycles as we do here, or if the dynamics is piecewise linear, as in exercise 17.3. As we have already argued in sect. 1.5.4 and appendix I.1.2, for nice hyperbolic flows whose symbolic dynamics is a subshift of finite type, the shadowing combinations *almost* cancel, and the spectral determinant is dominated by the fundamental cycles from (13.13), with longer cycles contributing only small "curvature" corrections.

These exact or nearly exact cancellations depend on the flow being smooth and the symbolic dynamics being a subshift of finite type. If the dynamics requires infinite Markov partition with pruning rules for longer and longer blocks, most of the shadowing combinations still cancel, but the few corresponding to the forbidden blocks do not, leading to a finite radius of convergence for the spectral determinant as in figure 13.5.

One striking aspect of the pruned cycle expansion (13.32) compared to the trace formulas such as (13.7) is that coefficients are not growing exponentially - indeed they all remain of order 1, so instead having a radius of convergence e^{-h}, in the example at hand the topological zeta function has the unit circle as the radius of convergence. In other words, exponentiating the spectral problem from a trace formula to a spectral determinant as in (13.19) increases the *analyticity domain*: the pole in the trace (13.8) at $z = e^{-h}$ is promoted to a smooth zero of the spectral determinant with a larger radius of convergence.

A detailed discussion of the radius of convergence is given in appendix D.1.

The very sensitive dependence of spectral determinants on whether the symbolic dynamics is or is not a subshift of finite type is the bad news that we should announce already now. If the system is generic and not structurally stable (see sect. 11.3), a smooth parameter variation is in no sense a smooth variation of topological dynamics - infinities of periodic orbits are created or destroyed, Markov graphs go from being finite to infinite and back. That will imply that the global averages that we intend to compute are generically nowhere differentiable functions of the system parameters, and averaging over families of dynamical systems can be a highly nontrivial enterprise; a simple illustration is the parameter dependence of the diffusion constant computed in a remark in chapter 23.

You might well ask: What is wrong with computing the entropy from (13.1)? Does all this theory buy us anything? An answer: If we count

K_n level by level, we ignore the self-similarity of the pruned tree - examine for example figure 10.13, or the cycle expansion of (13.26) - and the finite estimates of $h_n = \ln K_n/n$ converge nonuniformly to h, and on top of that with a slow rate of convergence, $|h - h_n| \approx O(1/n)$ as in (13.4). The determinant (13.9) is much smarter, as by construction it encodes the self-similarity of the dynamics, and yields the asymptotic value of h with no need for any finite n extrapolations.

So, the main lesson of learning how to count well, a lesson that will be affirmed over and over, is that while the trace formulas are a conceptually essential step in deriving and understanding periodic orbit theory, the spectral determinant is the right object to use in actual computations. Instead of resumming all of the exponentially many periodic points required by trace formulas at each level of truncation, spectral determinants incorporate only the small incremental corrections to what is already known - and that makes them more convergent and economical to use.

Résumé

What have we accomplished? We have related the number of topologically distinct paths from "this region" to "that region" in a chaotic system to the leading eigenvalue of the transition matrix T. The eigenspectrum of T is given by a certain sum over traces $\operatorname{tr} T^n$, and in this way the periodic orbit theory has entered the arena, already at the level of the topological dynamics, the crudest description of dynamics.

The main result of this chapter is the cycle expansion (13.21) of the topological zeta function (i.e., the spectral determinant of the transition matrix):

$$1/\zeta_{\mathrm{top}}(z) = 1 - \sum_{k=1} \hat{c}_k z^k .$$

For subshifts of finite type, the transition matrix is finite, and the topological zeta function is a finite polynomial evaluated by the loop expansion (13.13) of $\det(1 - zT)$. For infinite grammars the topological zeta function is defined by its cycle expansion. The topological entropy h is given by the smallest zero $z = e^{-h}$. This expression for the entropy is *exact*; in contrast to the definition (13.1), no $n \to \infty$ extrapolations of $\ln K_n/n$ are required.

Historically, these topological zeta functions were the inspiration for applying the transfer matrix methods of statistical mechanics to the problem of computation of dynamical averages for chaotic flows. The key result was the dynamical zeta function to be derived in chapter 16, a weighted generalization of the topological zeta function.

Contrary to claims one sometimes encounters in the literature, "exponential proliferation of trajectories" is not the problem; what limits the convergence

of cycle expansions is the proliferation of the grammar rules, or the "algorithmic complexity", as illustrated by sect. 13.6, and figure 13.5 in particular.

Commentary

Remark 13.1 "Entropy." The ease with which the topological entropy can be motivated obscures the fact that our definition does not lead to an invariant characterization of the dynamics, as the choice of symbolic dynamics is largely arbitrary: the same caveat applies to other entropies to be discussed in chapter 21, and to get proper invariants one needs to evaluate a supremum over all possible partitions. The key mathematical point that eliminates the need of such search is the existence of *generators*, i.e., partitions that under dynamics are able to probe the whole state space on arbitrarily small scales: more precisely a generator is a finite partition $\Omega = \omega_1 \ldots \omega_N$, with the following property: take $\mathcal{M}$ the subalgebra of the state space generated by Ω, and consider the partition built upon all possible intersections of sets $\phi^k(\beta_i)$, where ϕ is dynamical evolution, β_i is an element of $\mathcal{M}$ and k takes all possible integer values (positive as well as negative), then the closure of such a partition coincides with the algebra of all measurable sets. For a thorough (and readable) discussion of generators and how they allow a computation of the Kolmogorov entropy, see ref. [13.1] and chapter 21.

Remark 13.2 Perron-Frobenius matrices. For a proof of Perron theorem on the leading eigenvalue see ref. [1.21]. Sect. A4.1 of ref. [13.2] offers a clear discussion of the spectrum of the transition matrix.

Remark 13.3 Determinant of a graph. Many textbooks offer derivations of the loop expansions of characteristic polynomials for transition matrices and their Markov graphs, see for example refs. [13.3, 13.4, 13.5].

Remark 13.4 T is not trace class. Note to the erudite reader: the transition matrix T (in the infinite partition limit (13.19)) is *not* trace class in the sense of appendix J. Still the trace is well defined in the $n \to \infty$ limit.

Remark 13.5 Artin-Mazur zeta functions. Motivated by A. Weil's zeta function for the Frobenius map [13.6], Artin and Mazur [17.13] introduced the zeta function (13.21) that counts periodic points for diffeomorphisms (see also ref. [13.7] for their evaluation for maps of the interval). Smale [13.8] conjectured rationality of the zeta functions for Axiom A diffeomorphisms, later proved by Guckenheimer [13.9] and Manning [13.10]. See remark 17.4 on page 332 for more zeta function history.

Remark 13.6 Ordering periodic orbit expansions. In sect. 18.5 we will introduce an alternative way of hierarchically organizing cumulant expansions, in which the order is dictated by stability rather than cycle length: such a procedure may be better suited to perform computations when the symbolic dynamics is not well understood.

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