Exercises

Exercise 15.1 How unstable is the Hénon attractor?

(a) *Evaluate numerically the Lyapunov exponent λ by iterating the Hénon map*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 - ax^2 + y \\ bx \end{bmatrix}$$

for $a = 1.4$, $b = 0.3$.

(b) *Now check how robust is the Lyapunov exponent for the Hénon attractor? Evaluate numerically the Lyapunov exponent by iterating the Hénon map for $a = 1.39945219$, $b = 0.3$. How much do you trust now your result for the part (a) of this exercise?*

Exercise 15.2 Expectation value of a vector observable and its moments. *Check and extend the expectation value formulas (15.12) by evaluating the derivatives of $s(\beta)$ up to 4-th order for the space average $\langle \exp(\beta \cdot A^t) \rangle$ with a_i a vector quantity:*

(a)

$$\left.\frac{\partial s}{\partial \beta_i}\right|_{\beta=0} = \lim_{t\to\infty} \frac{1}{t} \langle A_i^t \rangle = \langle a_i \rangle \ , \tag{15.39}$$

(b)

$$\begin{aligned} \left.\frac{\partial^2 s}{\partial \beta_i \partial \beta_j}\right|_{\beta=0} &= \lim_{t\to\infty} \frac{1}{t} \left(\langle A_i^t A_j^t \rangle - \langle A_i^t \rangle \langle A_j^t \rangle \right) \\ &= \lim_{t\to\infty} \frac{1}{t} \langle (A_i^t - t\langle a_i \rangle)(A_j^t - t\langle a_j \rangle) \rangle \ . \end{aligned} \tag{15.40}$$

Note that the formalism is smart: it automatically yields the variance *from the mean, rather than simply the 2nd moment $\langle a^2 \rangle$.*

(c) *compute the third derivative of $s(\beta)$.*

(d) *compute the fourth derivative assuming that the mean in (15.39) vanishes, $\langle a_i \rangle = 0$. The 4-th order moment formula*

$$K(t) = \frac{\langle x^4(t) \rangle}{\langle x^2(t) \rangle^2} - 3 \tag{15.41}$$

that you have derived is known as kurtosis*: it measures a deviation from what the 4-th order moment would be were the distribution a pure Gaussian (see (23.22) for a concrete example). If the observable is a vector, the kurtosis is given by*

$$K(t) = \frac{\sum_{ij} \left[\langle A_i A_i A_j A_j \rangle + 2 \left(\langle A_i A_j \rangle \langle A_j A_i \rangle - \langle A_i A_i \rangle \langle A_j A_j \rangle \right) \right]}{\left(\sum_i \langle A_i A_i \rangle \right)^2} \tag{15.42}$$

Exercise 15.3 Pinball escape rate from numerical simulation*. *Estimate the escape rate for $R : a = 6$ 3-disk pinball by shooting 100,000 randomly initiated pinballs into the 3-disk system and plotting the logarithm of the number of trapped orbits as function of time. For comparison, a numerical simulation of ref. [8.2] yields $\gamma = .410\ldots$.*

Exercise 15.4 Rössler attractor Lyapunov exponents.

(a) *Evaluate numerically the expanding Lyapunov exponent λ_e of the Rössler attractor (2.17).*

(b) *Plot your own version of figure 15.6. Do not worry if it looks different, as long as you understand why your plot looks the way it does. (Remember the nonuniform contraction/expansion of figure 4.3.)*

(c) *Give your best estimate of λ_e. The literature gives surprisingly inaccurate estimates - see whether you can do better.*

(d) *Estimate the contracting Lyapunov exponent λ_c. Even though it is much smaller than λ_e, a glance at the stability matrix (4.4) suggests that you can probably get it by integrating the infinitesimal volume along a long-time trajectory, as in (4.47).*