Exercises

Exercise 8.1 **A pinball simulator.** *Implement the disk → disk maps to compute a trajectory of a pinball for a given starting point, and a given $R{:}a$ = (center-to-center distance):(disk radius) ratio for a 3-disk system. As this requires only computation of intersections of lines and circles together with specular reflections, implementation should be within reach of a high-school student. Please start working on this program now; it will be continually expanded in chapters to come, incorporating the Jacobian calculations, Newton root–finding, and so on.*

Fast code will use elementary geometry (only one $\sqrt{\cdots}$ per iteration, rest are multiplications) and eschew trigonometric functions. Provide a graphic display of the trajectories and of the Poincaré section iterates. To be able to compare with the numerical results of coming chapters, work with $R{:}a = 6$ and/or 2.5 values. Draw the correct versions of figure 1.10 or figure 10.4 for $R{:}a = 2.5$ and/or 6.

Exercise 8.2 **Trapped orbits.** *Shoot 100,000 trajectories from one of the disks, and trace out the strips of figure 1.10 for various $R{:}a$ by color coding the initial points in the Poincaré section by the number of bounces preceding their escape. Try also $R{:}a = 6{:}1$, though that might be too thin and require some magnification. The initial conditions can be randomly chosen, but need not - actually a clearer picture is obtained by systematic scan through regions of interest.*

Exercise 8.3 **Pinball stability.** *Add to your exercise 8.1 pinball simulator a routine that computes the the [2×2] Jacobian matrix. To be able to compare with the numerical results of coming chapters, work with $R{:}a = 6$ and/or 2.5 values.*

Exercise 8.4 **Stadium billiard.** *Consider the* Bunimovich stadium *[8.8, 8.9] defined in figure 8.1. The fundamental matrix associated with the reflection is given by (8.10). Here we take $\rho_k = -1$ for the semicircle sections of the boundary, and $\cos\phi_k$ remains constant for all bounces in a rotation sequence. The time of flight between two semicircle bounces is $\tau_k = 2\cos\phi_k$. The fundamental matrix of one semicircle reflection folowed by the flight to the next bounce is*

$$\begin{aligned}\mathbf{J} &= (-1)\begin{pmatrix} 1 & 2\cos\phi_k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2/\cos\phi_k & 1 \end{pmatrix} \\ &= (-1)\begin{pmatrix} -3 & 2\cos\phi_k \\ 2/\cos\phi_k & 1 \end{pmatrix}.\end{aligned}$$

*A shift must always be followed by $k = 1, 2, 3, \cdots$ bounces along a semicircle, hence the natural symbolic dynamics for this problem is n-ary, with the corresponding fundamental matrix given by shear (*ie. *the eigenvalues remain equal to 1 throughout the whole rotation), and k bounces inside a circle lead to*

$$\mathbf{J}^k = (-1)^k \begin{pmatrix} -2k-1 & 2k\cos\phi \\ 2k/\cos\phi & 2k-1 \end{pmatrix}. \tag{8.13}$$

The fundamental matrix of a cycle p of length n_p is given by

$$\mathbf{J}_p = (-1)^{\sum n_k} \prod_{k=1}^{n_p} \begin{pmatrix} 1 & \tau_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ n_k r_k & 1 \end{pmatrix} . \tag{8.14}$$

Adopt your pinball simulator to the Bunimovich stadium.

Exercise 8.5 **A test of your pinball simulator.** *Test your exercise 8.3 pinball simulator by computing numerically cycle stabilities by tracking distances to nearby orbits. Compare your result with the exact analytic formulas of exercise 9.3 and 9.4.*

Exercise 8.6 **Birkhoff coordinates.** *Prove that the Birkhoff coordinates are phase space volume preserving. Hint: compute the determinant of (8.11).*