## Exercises

**Exercise 12.1 Cycles of the Ulam map.** *Test your cycle-searching routines by computing a bunch of short cycles and their stabilities for the Ulam map*

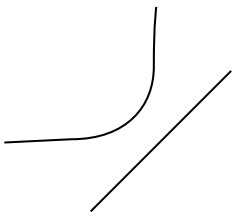
$$f(x) = 4x(1-x)\,. \tag{12.22}$$

**Exercise 12.2 Cycles stabilities for the Ulam map, exact.** *In exercise 12.1 you should have observed that the numerical results for the cycle stability eigenvalues (4.50) are exceptionally simple: the stability eigenvalue of the $x_0 = 0$ fixed point is 4, while the eigenvalue of any other $n$-cycle is $\pm 2^n$. Prove this. (Hint: the Ulam map can be conjugated to the tent map (10.6). This problem is perhaps too hard, but give it a try - the answer is in many introductory books on nonlinear dynamics.)*

**Exercise 12.3 Stability of billiard cycles.** *Compute stabilities of few simple cycles.*

**(a)** *A simple scattering billiard is the two-disk billiard. It consists of a disk of radius one centered at the origin and another disk of unit radius located at $L+2$. Find all periodic orbits for this system and compute their stabilities. (You might have done this already in exercise 1.2; at least now you will be able to see where you went wrong when you knew nothing about cycles and their extraction.)*

**(b)** *Find all periodic orbits and stabilities for a billiard ball bouncing between the diagonal $y = x$ and one of the hyperbola branches $y = -1/x$.*



**Exercise 12.4 Cycle stability.** *Add to the pinball simulator of exercise 8.1 a routine that evaluates the expanding eigenvalue for a given cycle.*

**Exercise 12.5 Pinball cycles.** *Determine the stability and length of all fundamental domain prime cycles of the binary symbol string lengths up to 5 (*or longer*) for $R : a = 6$ 3-disk pinball.*

**Exercise 12.6 Newton-Raphson method.** *Implement the Newton-Raphson method in 2-$d$ and apply it to determination of pinball cycles.*

**Exercise 12.7 Rössler system cycles.** *(continuation of exercise 4.4) Determine all cycles up to 5 Poincaré sections returns for the Rössler system (2.17), as well as their stabilities. (Hint: implement (12.16), the multipoint shooting methods for flows; you can cross-check your shortest cycles against the ones listed in table 12.1.)*

| $n_p$ | $p$ | $y_p$ | $z_p$ | $\Lambda_e$ |
|---|---|---|---|---|
| 1 | 1 | 6.091768319056803 | 1.299731937639821 | -2.4039535318268 |
| 2 | 01 | 3.915804049621049 | 3.692833386542665 | -3.5120069815161 |
| 3 | 001 | 2.278281031720258 | 7.416480984019008 | -2.3419235232340 |
| | 011 | 2.932877559129124 | 5.670805943881501 | 5.3449081538885 |
| 4 | 0111 | 3.466758713211455 | 4.506217531477667 | -16.6967406980700 |
| 5 | 01011 | 4.162798782914948 | 3.303903338609633 | -23.1995830097831 |
| | 01111 | 3.278914359770783 | 4.890452922955567 | 36.8863297988981 |
| 6 | 001011 | 2.122093931936202 | 7.886172854283211 | -6.8576654190825 |
| | 010111 | 4.059210605826523 | 3.462265228606606 | 61.6490940089068 |
| | 011111 | 3.361494458061049 | 4.718206217035575 | -92.0825560711089 |
| 7 | 0101011 | 3.842769382372052 | 3.815493592299824 | 77.7611048852412 |
| | 0110111 | 3.025956697151134 | 5.451444475664179 | -95.1838846735358 |
| | 0101111 | 4.102255295518855 | 3.395643547170646 | -142.2379888163439 |
| | 0111111 | 3.327986189581191 | 4.787462810306583 | 218.0283602810993 |

**Table 12.1:** The Rössler system (2.17): The itinerary $p$, a periodic point $x_p = (0, y_p, z_p)$ and the expanding eigenvalue $\Lambda_p$ for all cycles up to the topological length 7. (Joachim Mathiesen, Gabor Simon, Arindam Basu)

**Exercise 12.8 Cycle stability, helium.** *Add to the helium integrator of exercise 2.10 a routine that evaluates the expanding eigenvalue for a given cycle.*

**Exercise 12.9 Colinear helium cycles.** *Determine the stability and length of all fundamental domain prime cycles up to symbol sequence length 5 or longer for collinear helium of figure 35.5.*

**Exercise 12.10 Uniqueness of unstable cycles$^{***}$.** *Prove that there exists only one 3-disk prime cycle for a given finite admissible prime cycle symbol string. Hints: look at the Poincaré section mappings; can you show that there is exponential contraction to a unique periodic point with a given itinerary? Exercise 26.1 might be helpful in this effort.*

**Exercise 12.11 Inverse iteration method for a Hamiltonian repeller.** *Consider the Hénon map (3.15) for area-preserving ("Hamiltonian") parameter value $b = -1$. The coordinates of a periodic orbit of length $n_p$ satisfy the equation*

$$x_{p,i+1} + x_{p,i-1} = 1 - a x_{p,i}^2 , \quad i = 1, ..., n_p , \tag{12.23}$$

*with the periodic boundary condition $x_{p,0} = x_{p,n_p}$. Verify that the itineraries and the stabilities of the short periodic orbits for the Hénon repeller (12.23) at $a = 6$ are as listed in table 12.2.*

**Hint**: *you can use any cycle-searching routine you wish, but for the complete repeller case (all binary sequences are realized), the cycles can be evaluated simply by inverse iteration, using the inverse of (12.23)*

$$x''_{p,i} = S_{p,i} \sqrt{\frac{1 - x'_{p,i+1} - x'_{p,i-1}}{a}}, \quad i = 1, ..., n_p .$$

*Here $S_{p,i}$ are the signs of the corresponding cycle point coordinates, $S_{p,i} = x_{p,i}/|x_{p,i}|$.*

*(G. Vattay)*

**Exercise 12.12 "Center of mass" puzzle$^{**}$.** *Why is the "center of mass", listed in table 12.2, a simple rational every so often?*

| $p$ | $\Lambda_p$ | $\sum x_{p,i}$ |
|---|---|---|
| 0 | $0.71516752438\times10^{1}$ | -0.6076252185107 |
| 1 | $-0.29528463259\times10^{1}$ | 0.2742918851774 |
| 10 | $-0.98989794855\times10^{1}$ | 0.3333333333333 |
| 100 | $-0.13190727397\times10^{3}$ | -0.2060113295833 |
| 110 | $0.55896964996\times10^{2}$ | 0.5393446629166 |
| 1000 | $-0.10443010730\times10^{4}$ | -0.8164965809277 |
| 1100 | $0.57799826989\times10^{4}$ | 0.0000000000000 |
| 1110 | $-0.10368832509\times10^{3}$ | 0.8164965809277 |
| 10000 | $-0.76065343718\times10^{4}$ | -1.4260322065792 |
| 11000 | $0.44455240007\times10^{4}$ | -0.6066540777738 |
| 10100 | $0.77020248597\times10^{3}$ | 0.1513755016405 |
| 11100 | $-0.71068835616\times10^{3}$ | 0.2484632276044 |
| 11010 | $-0.58949885284\times10^{3}$ | 0.8706954728949 |
| 11110 | $0.39099424812\times10^{3}$ | 1.0954854155465 |
| 100000 | $-0.54574527060\times10^{5}$ | -2.0341342556665 |
| 110000 | $0.32222060985\times10^{5}$ | -1.2152504370215 |
| 101000 | $0.51376165109\times10^{4}$ | -0.4506624359329 |
| 111000 | $-0.47846146631\times10^{4}$ | -0.3660254037844 |
| 110100 | $-0.63939998436\times10^{4}$ | 0.3333333333333 |
| 101100 | $-0.63939998436\times10^{4}$ | 0.3333333333333 |
| 111100 | $0.39019387269\times10^{4}$ | 0.5485837703548 |
| 111010 | $0.10949094597\times10^{4}$ | 1.1514633582661 |
| 111110 | $-0.10433841694\times10^{4}$ | 1.3660254037844 |

**Table 12.2:** All periodic orbits up to 6 bounces for the Hamiltonian Hénon mapping (12.23) with $a = 6$. Listed are the cycle itinerary, its expanding eigenvalue $\Lambda_p$, and its "center of mass". The "center of mass" is listed because it turns out the "center of mass" is often a simple rational or a quadratic irrational.