Exercises

Exercise 17.1 **Escape rate for a 1-d repeller, numerically.** *Consider the quadratic map*

$$f(x) = Ax(1-x) \tag{17.31}$$

on the unit interval. The trajectory of a point starting in the unit interval either stays in the interval forever or after some iterate leaves the interval and diverges to minus infinity. Estimate numerically the escape rate (19.8), the rate of exponential decay of the measure of points remaining in the unit interval, for either $A = 9/2$ or $A = 6$. Remember to compare your numerical estimate with the solution of the continuation of this exercise, exercise 18.2.

Exercise 17.2 **Spectrum of the "golden mean" pruned map.** *(medium - Exercise 13.6 continued)*

(a) *Determine an expression for $\mathrm{tr}\,\mathcal{L}^n$, the trace of powers of the Perron-Frobenius operator (14.10) for the tent map of exercise 13.6.*

(b) *Show that the spectral determinant for the Perron-Frobenius operator is*

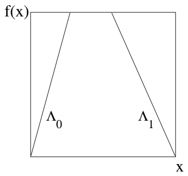
$$\begin{aligned}\det(1-z\mathcal{L}) \;=\; & \prod_{k\ \mathrm{even}}\left(1-\frac{z}{\Lambda^{k+1}}-\frac{z^2}{\Lambda^{2k+2}}\right) \\ & \prod_{k\ \mathrm{odd}}\left(1+\frac{z}{\Lambda^{k+1}}+\frac{z^2}{\Lambda^{2k+2}}\right). \end{aligned} \tag{17.32}$$

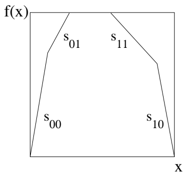
Exercise 17.3 **Dynamical zeta functions.** *(easy)*

(a) *Evaluate in closed form the dynamical zeta function*

$$1/\zeta(z) = \prod_p \left(1-\frac{z^{n_p}}{|\Lambda_p|}\right),$$

for the piecewise-linear map (15.17) with the left branch slope Λ_0, the right branch slope Λ_1.





(b) *What if there are four different slopes s_{00}, s_{01}, s_{10}, and s_{11} instead of just two, with the preimages of the gap adjusted so that junctions of branches s_{00}, s_{01} and s_{11}, s_{10} map in the gap in one iteration? What would the dynamical zeta function be?*

Exercise 17.4 **Dynamical zeta functions from Markov graphs.** *Extend sect. 13.3 to evaluation of dynamical zeta functions for piecewise linear maps with finite Markov graphs. This generalizes the results of exercise 17.3.*

Exercise 17.5 **Zeros of infinite products.** *Determination of the quantities of interest by periodic orbits involves working with infinite product formulas.*

(a) *Consider the infinite product*

$$F(z) = \prod_{k=0}^{\infty}(1+f_k(z))$$

where the functions f_k are "sufficiently nice." This infinite product can be converted into an infinite sum by the use of a logarithm. Use the properties of infinite sums to develop a sensible definition of infinite products.

(b) *If z^* is a root of the function F, show that the infinite product diverges when evaluated at z^*.*

(c) *How does one compute a root of a function represented as an infinite product?*

(d) *Let p be all prime cycles of the binary alphabet $\{0,1\}$. Apply your definition of $F(z)$ to the infinite product*

$$F(z) = \prod_p (1-\frac{z^{n_p}}{\Lambda^{n_p}})$$

(e) *Are the roots of the factors in the above product the zeros of $F(z)$?*

(Per Rosenqvist)

Exercise 17.6 **Dynamical zeta functions as ratios of spectral determinants.** *(medium) Show that the zeta function*

$$1/\zeta(z) = \exp\left(-\sum_p\sum_{r=1}\frac{1}{r}\frac{z^{n_p}}{|\Lambda_p|^r}\right)$$

can be written as the ratio $1/\zeta(z) = \det(1-z\mathcal{L}_{(0)})/\det(1-z\mathcal{L}_{(1)})$,
where $\det(1-z\mathcal{L}_{(s)}) = \prod_p\prod_{k=0}^{\infty}(1-z^{n_p}/|\Lambda_p|\Lambda_p^{k+s})$.

Exercise 17.7 **Contour integral for survival probability.** *Perform explicitly the contour integral appearing in (17.19).*

Exercise 17.8 Dynamical zeta function for maps. *In this problem we will compare the dynamical zeta function and the spectral determinant. Compute the exact dynamical zeta function for the skew Ulam tent map (14.45)*

$$1/\zeta(z) = \prod_{p\in P}\left(1 - \frac{z^{n_p}}{|\Lambda_p|}\right).$$

What are its roots? Do they agree with those computed in exercise 14.7?

Exercise 17.9 Dynamical zeta functions for Hamiltonian maps. *Starting from*

$$1/\zeta(s) = \exp\left(-\sum_p \sum_{r=1}^{\infty} \frac{1}{r} t_p^r\right)$$

for a 2-dimensional Hamiltonian map. Using the equality

$$1 = \frac{1}{(1-1/\Lambda)^2}(1 - 2/\Lambda + 1/\Lambda^2),$$

show that

$$1/\zeta = \det(1-\mathcal{L})\det(1-\mathcal{L}_{(2)})/\det(1-\mathcal{L}_{(1)})^2 .$$

In this expression $\det(1 - z\mathcal{L}_{(k)})$ *is the expansion one gets by replacing* $t_p \to t_p/\Lambda_p^k$ *in the spectral determinant.*

Exercise 17.10 Riemann ζ function. *The Riemann ζ function is defined as the sum*

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \qquad s \in \mathbb{C}.$$

(a) *Use factorization into primes to derive the Euler product representation*

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} .$$

The dynamical zeta function exercise 17.15 is called a "zeta" function because it shares the form of the Euler product representation with the Riemann zeta function.

(b) *(Not trivial:) For which complex values of s is the Riemann zeta sum convergent?*

(c) *Are the zeros of the terms in the product, $s = -\ln p$, also the zeros of the Riemann ζ function? If not, why not?*

Exercise 17.11 <u>Finite truncations.</u> *(easy) Suppose we have a one-dimensional system with complete binary dynamics, where the stability of each orbit is given by a simple multiplicative rule:*

$$\Lambda_p = \Lambda_0^{n_{p,0}}\Lambda_1^{n_{p,1}}, \qquad n_{p,0} = \#0 \text{ in } p\,,\; n_{p,1} = \#1 \text{ in } p\,,$$

so that, for example, $\Lambda_{00101} = \Lambda_0^3\Lambda_1^2$.

(a) *Compute the dynamical zeta function for this system; perhaps by creating a transfer matrix analogous to (15.19), with the right weights.*

(b) *Compute the finite p truncations of the cycle expansion, i.e. take the product only over the p up to given length with $n_p \le N$, and expand as a series in z*

$$\prod_p\left(1 - \frac{z^{n_p}}{|\Lambda_p|}\right).$$

Do they agree? If not, how does the disagreement depend on the truncation length N?