Exercises

The problems that you should do have **underlined titles**. The rest (**smaller type**) are optional. Difficult problems are marked by any number of *** stars.

Exercise 2.1 Trajectories do not intersect. *A trajectory in the state space $\mathcal{M}$ is the set of points one gets by evolving $x \in \mathcal{M}$ forwards and backwards in time:*

$$C_x = \{y \in \mathcal{M} : f^t(x) = y \text{ for } t \in \mathbb{R}\} .$$

Show that if two trajectories intersect, then they are the same curve.

Exercise 2.2 Evolution as a group. *The trajectory evolution f^t is a one-parameter semigroup, where (2.3)*

$$f^{t+s} = f^t \circ f^s .$$

Show that it is a commutative semigroup.

In this case, the commutative character of the (semi-)group of evolution functions comes from the commutative character of the time parameter under addition. Can you think of any other (semi-)group replacing time?

Exercise 2.3 Almost ODE's.

(a) *Consider the point x on $\mathbb{R}$ evolving according $\dot{x} = e^{\dot{x}}$. Is this an ordinary differential equation?*

(b) *Is $\dot{x} = x(x(t))$ an ordinary differential equation?*

(c) *What about $\dot{x} = x(t+1)$?*

Exercise 2.4 All equilibrium points are fixed points. *Show that a point of a vector field v where the velocity is zero is a fixed point of the dynamics f^t.*

Exercise 2.5 Gradient systems. *Gradient systems (or 'potential problems') are a simple class of dynamical systems for which the velocity field is given by the gradient of an auxiliary function, the 'potential' ϕ*

$$\dot{x} = -\nabla\phi(x)$$

where $x \in \mathbb{R}^d$, and ϕ is a function from that space to the reals $\mathbb{R}$.

(a) *Show that the velocity of the particle is in the direction of most rapid decrease of the function ϕ.*

(b) *Show that all extrema of ϕ are fixed points of the flow.*

(c) *Show that it takes an infinite amount of time for the system to reach an equilibrium point.*

(d) *Show that there are no periodic orbits in gradient systems.*

Exercise 2.6 **Runge-Kutta integration.** *Implement the fourth-order Runge-Kutta integration formula (see, for example, ref. [2.12]) for $\dot{x} = v(x)$:*

$$\begin{aligned} x_{n+1} &= x_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(\delta\tau^5) \\ k_1 &= \delta\tau\, v(x_n)\,, \qquad k_2 = \delta\tau\, v(x_n + k_1/2) \\ k_3 &= \delta\tau\, v(x_n + k_2/2) \\ k_4 &= \delta\tau\, v(x_n + k_3)\,. \end{aligned} \tag{2.20}$$

If you already know your Runge-Kutta, program what you believe to be a better numerical integration routine, and explain what is better about it.

Exercise 2.7 **Rössler flow.** *Use the result of exercise 2.6 or some other integration routine to integrate numerically the Rössler system (2.17). Does the result look like a 'strange attractor'?*

Exercise 2.8 **Equilibria of the Rössler system.**

(a) *Find all equilibrium points (x_q, y_q, z_q) of the Rössler system (2.17). How many are there?*

(b) *Assume that $b = a$. As we shall see, some surprisingly large, and surprisingly small numbers arise in this system. In order to understand their size, introduce parameters*

$$\epsilon = a/c\,,\ D = 1 - 4\epsilon^2\,,\ p^{\pm} = (1 \pm \sqrt{D})/2\,. \tag{2.21}$$

Express all the equilibria in terms of $(c, \epsilon, D, p^{\pm})$. Expand equilibria to the first order in ϵ. Note that it makes sense because for $a = b = 0.2$, $c = 5.7$ in (2.17), $\epsilon \approx 0.03$. (continued as exercise 3.1)

(Rytis Paškauskas)

Exercise 2.9 **Can you integrate me?** *Integrating equations numerically is not for the faint of heart. It is not always possible to establish that a set of nonlinear ordinary differential equations has a solution for all times and there are many cases were the solution only exists for a limited time interval, as, for example, for the equation $\dot{x} = x^2\,,\ x(0) = 1\,.$*

(a) *For what times do solutions of*

$$\dot{x} = x(x(t))$$

exist? Do you need a numerical routine to answer this question?

(b) *Let's test the integrator you wrote in exercise 2.6. The equation $\ddot{x} = -x$ with initial conditions $x(0) = 2$ and $\dot{x} = 0$ has as solution $x(t) = e^{-t}(1 + e^{2t})$. Can your integrator reproduce this solution for the interval $t \in [0, 10]$? Check you solution by plotting the error as compared to the exact result.*

(c) *Now we will try something a little harder. The equation is going to be third order*

$$\dddot{x} + 0.6\ddot{x} + \dot{x} - |x| + 1 = 0\,,$$

which can be checked–numerically–to be chaotic. As initial conditions we will always use $\ddot{x}(0) = \dot{x}(0) = x(0) = 0$. Can you reproduce the result $x(12) = 0.8462071873$ (all digits are significant)? Even though the equation being integrated is chaotic, the time intervals are not long enough for the exponential separation of trajectories to be noticeable (the exponential growth factor is ≈ 2.4).

(d) *Determine the time interval for which the solution of $\dot{x} = x^2, x(0) = 1$ exists.*

Exercise 2.10 **Classical collinear helium dynamics.** *In order to apply periodic orbit theory to quantization of helium we shall need to compute classical periodic orbits of the helium system. In this exercise we commence their evaluation for the collinear helium atom (6.6)*

$$H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - \frac{Z}{r_1} - \frac{Z}{r_2} + \frac{1}{r_1 + r_2}\,.$$

The nuclear charge for helium is $Z = 2$. Colinear helium has only 3 degrees of freedom and the dynamics can be visualized as a motion in the (r_1, r_2), $r_i \geq 0$ quadrant. In (r_1, r_2)-coordinates the potential is singular for $r_i \to 0$ nucleus-electron collisions. These 2-body collisions can be regularized by rescaling the coordinates, with details given in sect. 7.3. In the transformed coordinates (x_1, x_2, p_1, p_2) the Hamiltonian equations of motion take the form

$$\begin{aligned}
\dot{P}_1 &= 2Q_1\left[2 - \frac{P_2^2}{8} - Q_2^2(1 + \frac{Q_2^2}{R^4})\right] \\
\dot{P}_2 &= 2Q_2\left[2 - \frac{P_1^2}{8} - Q_1^2(1 + \frac{Q_1^2}{R^4})\right] \\
\dot{Q}_1 &= \frac{1}{4}P_1Q_2^2\,, \quad \dot{Q}_2 = \frac{1}{4}P_2Q_1^2\,.
\end{aligned} \tag{2.22}$$

where $R = (Q_1^2 + Q_2^2)^{1/2}$.

(a) *Integrate the equations of motion by the fourth order Runge-Kutta computer routine of exercise 2.6 (or whatever integration routine you like). A convenient way to visualize the 3-d state space orbit is by projecting it onto the 2-dimensional $(r_1(t), r_2(t))$ plane. (continued as exercise 3.4)*

(Gregor Tanner, Per Rosenqvist)