Exercises

Exercise 10.1 **Binary symbolic dynamics.** *Verify that the shortest prime binary cycles of the unimodal repeller of figure 10.6 are* $\overline{0}$, $\overline{1}$, $\overline{01}$, $\overline{001}$, $\overline{011}$, $\cdots$. *Compare with table 10.1. Try to sketch them in the graph of the unimodal function* $f(x)$*; compare ordering of the periodic points with figure 10.9. The point is that while overlayed on each other the longer cycles look like a hopeless jumble, the cycle points are clearly and logically ordered by the alternating binary tree.*

Exercise 10.2 **Generating prime cycles.** *Write a program that generates all binary prime cycles up to given finite length.*

Exercise 10.3 **A contracting baker's map.** *Consider a contracting (or "dissipative") baker's defined in exercise 4.6.*

The symbolic dynamics encoding of trajectories is realized via symbols 0 ($y \leq 1/2$) *and* 1 ($y > 1/2$). *Consider the observable* $a(x, y) = x$. *Verify that for any periodic orbit* p ($\epsilon_1 \ldots \epsilon_{n_p}$), $\epsilon_i \in \{0, 1\}$

$$A_p = \frac{3}{4} \sum_{j=1}^{n_p} \delta_{j,1} \,.$$

Exercise 10.4 **Unimodal map symbolic dynamics.** *Show that the tent map point* $\gamma(S^+)$ *with future itinerary* S^+ *is given by converting the sequence of* s_n*'s into a binary number by the algorithm (10.9). This follows by inspection from the binary tree of figure 10.9.*

Exercise 10.5 **Unimodal map kneading value.** *Consider the 1-d quadratic map*

$$f(x) = Ax(1-x)\,, \qquad A = 3.8\,. \tag{10.23}$$

(a) *(easy) Plot (10.23), and the first 4-8 (whatever looks better) iterates of the critical point* $x_c = 1/2$.

(b) *(hard) Draw corresponding intervals of the partition of the unit interval as levels of a Cantor set, as in the symbolic dynamics partition of figure 10.6 (b). Note, however, that some of the intervals of figure 10.6 (b) do not appear in this case - they are* pruned.

(c) *(medium) Produce ChaosBook.org quality figure 10.6 (a).*

(d) *(easy) Check numerically that* $K = S^+(x_c)$*, the itinerary or the "kneading sequence" of the critical point is*

$$K = 10110111101101111010111101111110\ldots$$

The tent map point $\gamma(S^+)$ *with future itinerary* S^+ *is given by converting the sequence of* s_n*'s into a binary number by the algorithm (10.9),*

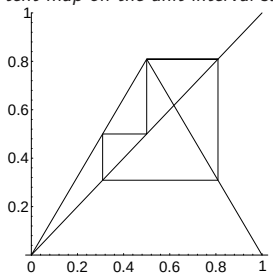
$$\begin{aligned} w_{n+1} &= \begin{cases} w_n & \text{if } s_{n+1} = 0 \\ 1 - w_n & \text{if } s_{n+1} = 1 \end{cases}, \qquad w_1 = s_1 \\ \gamma(S^+) &= 0.w_1 w_2 w_3 \ldots = \sum_{n=1}^{\infty} w_n / 2^n \,. \end{aligned}$$

(e) *(medium) List the the corresponding kneading value (10.10) sequence* $\kappa = \gamma(K)$ *to the same number of digits as* K.

(f) *(hard) Plot the missing dike map, figure 10.10, in ChaosBook.org quality, with the same kneading sequence* K *as* $f(x)$. *The dike map is obtained by slicing off all* $\gamma\left(S^+(x_0)\right) > \kappa$*, from the complete tent map figure 10.6 (a), see (10.11).*

How this kneading sequence is converted into a series of pruning rules is a dark art, relegated to sect. 13.6 and appendix D.1.

Exercise 10.6 **"Golden mean" pruned map.** *Consider a symmetrical tent map on the unit interval such that its highest point belongs to a 3-cycle:*



(a) *Find the absolute value* Λ *for the slope (the two different slopes* $\pm\Lambda$ *just differ by a sign) where the maximum at* $1/2$ *is part of a period three orbit, as in the figure.*

(b) *Show that no orbit of this map can visit the region* $x > (1+\sqrt{5})/4$ *more than once. Verify that once an orbit exceeds* $x > (\sqrt{5}-1)/4$*, it does not reenter the region* $x < (\sqrt{5}-1)/4$.

(c) *If an orbit is in the interval* $(\sqrt{5}-1)/4 < x < 1/2$*, where will it be on the next iteration?*

(d) *If the symbolic dynamics is such that for* $x < 1/2$ *we use the symbol 0 and for* $x > 1/2$ *we use the symbol 1, show that no periodic orbit will have the substring* _00_ *in it.*

(e) *On the second thought, is there a periodic orbit that violates the above* $_00_$ *pruning rule?*

For continuation, see exercise 13.6 and exercise 17.2. See also exercise 13.7 and exercise 13.8.

Exercise 10.7 Binary 3-step transition matrix. *Construct [8×8] binary 3-step transition matrix analogous to the 2-step transition matrix (10.13). Convince yourself that the number of terms of contributing to* $\mathrm{tr}\, T^n$ *is independent of the memory length, and that this [*$2^m \times 2^m$*] trace is well defined in the infinite memory limit* $m \to \infty$.