## Exercises

**Exercise 3.1** **Poincaré sections of the Rössler flow.** *(continuation of exercise 2.8) Calculate numerically a Poincaré section (or several Poincaré sections) of the Rössler flow. As the Rössler flow state space is 3-dimensional, the flow maps onto a 2-dimensional Poincaré section. Do you see that in your numerical results? How good an approximation would a replacement of the return map for this section by a 1-dimensional map be? More precisely, estimate the thickness of the strange attractor. (continued as exercise 4.4)*

*(Rytis Paškauskas)*

**Exercise 3.2** **A return Poincaré map for the Rössler flow.** *(continuation of exercise 3.1) That Poincaré return maps of figure 3.3 appear multimodal and non-invertible is an artifact of projections of a 2-dimensional return map $(R_n, z_n) \to (R_{n+1}, z_{n+1})$ onto a 1-dimensional subspace $R_n \to R_{n+1}$.*

*Construct a genuine $s_{n+1} = f(s_n)$ return map by parametrazing points on a Poincaré section of the attractor figure 3.2 by a Euclidean length $s$ computed curvilinearly along the attractor section.*

*This is best done (using methods to be developed in what follows) by a continuation of the unstable manifold of the 1-cycle embedded in the strange attractor, figure 12.1 (b).*

*(Predrag Cvitanović)*

**Exercise 3.3** **Arbitrary Poincaré sections.** *We will generalize the construction of Poincaré sections so that they can have any shape, as specified by the equation $U(x) = 0$.*

**(a)** *Start by modifying your integrator so that you can change the coordinates once you get near the Poincaré section. You can do this easily by writing the equations as*

$$\frac{dx_k}{ds} = \kappa f_k \,, \tag{3.19}$$

*with $dt/ds = \kappa$, and choosing $\kappa$ to be $1$ or $1/f_1$. This allows one to switch between $t$ and $x_1$ as the integration 'time.'*

**(b)** *Introduce an extra dimension $x_{n+1}$ into your system and set*

$$x_{n+1} = U(x) \,. \tag{3.20}$$

*How can this be used to find a Poincaré section?*

**Exercise 3.4** **Classical collinear helium dynamics.** *(continuation of exercise 2.10)*

*Make a Poincaré surface of section by plotting $(r_1, p_1)$ whenever $r_2 = 0$: Note that for $r_2 = 0$, $p_2$ is already determined by (6.6). Compare your results with figure 35.3 (b).*

*(Gregor Tanner, Per Rosenqvist)*

**Exercise 3.5** **Hénon map fixed points.** *Show that the two fixed points* $(x_0, x_0)$, $(x_1, x_1)$ *of the Hénon map (3.15) are given by*

$$\begin{aligned} x_0 &= \frac{-(1-b) - \sqrt{(1-b)^2 + 4a}}{2a}\,, \\ x_1 &= \frac{-(1-b) + \sqrt{(1-b)^2 + 4a}}{2a}\,. \end{aligned} \tag{3.21}$$

**Exercise 3.6** **How strange is the Hénon attractor?**

**(a)** *Iterate numerically some 100,000 times or so the Hénon map*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 - ax^2 + y \\ bx \end{bmatrix}$$

*for* $a = 1.4$, $b = 0.3$ . *Would you describe the result as a 'strange attractor'? Why?*

**(b)** *Now check how robust the Hénon attractor is by iterating a slightly different Hénon map, with* $a = 1.39945219$, $b = 0.3$. *Keep at it until the 'strange' attractor vanishes like the smile of the Chesire cat. What replaces it? Would you describe the result as a 'strange attractor'? Do you still have confidence in your own claim for the part (a) of this exercise?*

**Exercise 3.7** **Fixed points of maps.** *A continuous function* $F$ *is a contraction of the unit interval if it maps the interval inside itself.*

**(a)** *Use the continuity of* $F$ *to show that a one-dimensional contraction* $F$ *of the interval* $[0, 1]$ *has at least one fixed point.*

**(b)** *In a uniform (hyperbolic) contraction the slope of* $F$ *is always smaller than one,* $|F'| < 1$. *Is the composition of uniform contractions a contraction? Is it uniform?*