Exercises

Exercise 14.1 **Integrating over Dirac delta functions.** *Let us verify a few of the properties of the delta function and check (14.9), as well as the formulas (14.7) and (14.8) to be used later.*

(a) *If $f : \mathbb{R}^d \to \mathbb{R}^d$, show that*

$$\int_{\mathbb{R}^d} dx\, \delta\left(f(x)\right) = \sum_{x \in f^{-1}(0)} \frac{1}{|\det \partial_x f|}\,.$$

(b) *The delta function can be approximated by a sequence of Gaussians*

$$\int dx\, \delta(x) f(x) = \lim_{\sigma \to 0} \int dx\, \frac{e^{-\frac{x^2}{2\sigma}}}{\sqrt{2\pi\sigma}} f(x)\,.$$

Use this approximation to see whether the formal expression

$$\int_{\mathbb{R}} dx\, \delta(x^2)$$

makes sense.

Exercise 14.2 **Derivatives of Dirac delta functions.** *Consider $\delta^{(k)}(x) = \frac{\partial^k}{\partial x^k}\delta(x)$.*

Using integration by parts, determine the value of

$$\int_{\mathbb{R}} dx\, \delta'(y) \quad , \qquad \text{where } y = f(x) - x \tag{14.41}$$

$$\int dx\, \delta^{(2)}(y) = \sum_{\{x : y(x)=0\}} \frac{1}{|y'|} \left\{ 3\frac{(y'')^2}{(y')^4} - \frac{y'''}{(y')^3} \right\} \tag{14.42}$$

$$\int dx\, b(x) \delta^{(2)}(y) = \sum_{\{x : y(x)=0\}} \frac{1}{|y'|} \left\{ \frac{b''}{(y')^2} - \frac{b' y''}{(y')^3} + b \left(3\frac{(y'')^2}{(y')^4} - \frac{y'''}{(y')^3} \right) \right\}. \tag{14.43}$$

These formulas are useful for computing effects of weak noise on deterministic dynamics [14.5].

Exercise 14.3 **$\mathcal{L}^t$ generates a semigroup.** *Check that the Perron-Frobenius operator has the semigroup property,*

$$\int_M dz \mathcal{L}^{t_2}(y,z)\, \mathcal{L}^{t_1}(z,x) = \mathcal{L}^{t_2+t_1}(y,x)\,, \qquad t_1, t_2 \geq 0\,. \tag{14.44}$$

As the flows in which we tend to be interested are invertible, the $\mathcal{L}$'s that we will use often do form a group, with $t_1, t_2 \in \mathbb{R}$.

Exercise 14.4 Escape rate of the tent map.

(a) *Calculate by numerical experimentation the log of the fraction of trajectories remaining trapped in the interval* $[0, 1]$ *for the tent map*

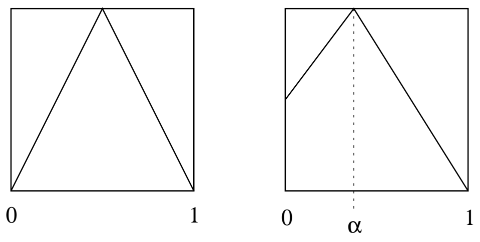
$$f(x) = a(1 - 2|x - 0.5|)$$

for several values of a.

(b) *Determine analytically the* a *dependence of the escape rate* $\gamma(a)$.

(c) *Compare your results for (a) and (b).*

Exercise 14.5 Invariant measure. *We will compute the invariant measure for two different piecewise linear maps.*



(a) *Verify the matrix* $\mathcal{L}$ *representation (15.19).*

(b) *The maximum value of the first map is 1. Compute an invariant measure for this map.*

(c) *Compute the leading eigenvalue of* $\mathcal{L}$ *for this map.*

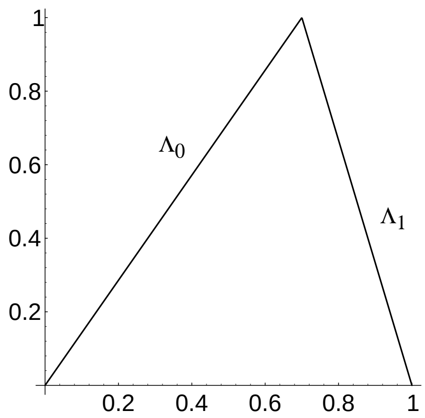
(d) *For this map there is an infinite number of invariant measures, but only one of them will be found when one carries out a numerical simulation. Determine that measure, and explain why your choice is the natural measure for this map.*

(e) *In the second map the maximum occurs at* $\alpha = (3-\sqrt{5})/2$ *and the slopes are* $\pm(\sqrt{5}+1)/2$. *Find the natural measure for this map. Show that it is piecewise linear and that the ratio of its two values is* $(\sqrt{5}+1)/2$.

(medium difficulty)

Exercise 14.6 Escape rate for a flow conserving map. *Adjust* Λ_0, Λ_1 *in (15.17) so that the gap between the intervals* $\mathcal{M}_0$, $\mathcal{M}_1$ *vanishes. Show that the escape rate equals zero in this situation.*

Exercise 14.7 Eigenvalues of the Perron-Frobenius operator for the skew Ulam tent map. *Show that for the skew Ulam tent map*



$$f(x) = \begin{cases} f_0(x) = \Lambda_0 x\,, & x \in \mathcal{M}_0 = [0, 1/\Lambda_0) \\ f_1(x) = \frac{\Lambda_0}{\Lambda_0 - 1}(1-x)\,, & x \in \mathcal{M}_1 = (1/\Lambda_0, 1]\,. \end{cases} \tag{14.45}$$

the eigenvalues are available analytically, compute the first few.

Exercise 14.8 "Kissing disks"* *(continuation of exercises 8.1 and 8.2). Close off the escape by setting $R = 2$, and look in real time at the density of the Poincaré section iterates for a trajectory with a randomly chosen initial condition. Does it look uniform? Should it be uniform? (Hint - phase space volumes are preserved for Hamiltonian flows by the Liouville theorem). Do you notice the trajectories that loiter near special regions of phase space for long times? These exemplify "intermittency", a bit of unpleasantness to which we shall return in chapter 22.*

Exercise 14.9 Invariant measure for the Gauss map. *Consider the Gauss map (we shall need this map in chapter 27):*

$$f(x) = \begin{cases} \frac{1}{x} - \left[\frac{1}{x}\right] & x \neq 0 \\ 0 & x = 0 \end{cases}$$

where [ ] denotes the integer part.

(a) *Verify that the density*

$$\rho(x) = \frac{1}{\log 2} \frac{1}{1+x}$$

is an invariant measure for the map.

(b) *Is it the natural measure?*

Exercise 14.10 $\mathcal{A}$ as a generator of translations. *Verify that for a constant velocity field the evolution generator $\mathcal{A}$ in (14.30) is the generator of translations,*

$$e^{tv\frac{\partial}{\partial x}} a(x) = a(x + tv)\,.$$

Exercise 14.11 Incompressible flows. *Show that (14.9) implies that $\rho_0(x) = 1$ is an eigenfunction of a volume-preserving flow with eigenvalue $s_0 = 0$. In particular, this implies that the natural measure of hyperbolic and mixing Hamiltonian flows is uniform. Compare this results with the numerical experiment of exercise 14.8.*