Exercises

Exercise 6.1 Complex nonlinear Schrödinger equation. *Consider the complex nonlinear Schrödinger equation in one spatial dimension [6.1]:*

$$i\frac{\partial\phi}{\partial t}+\frac{\partial^2\phi}{\partial x^2}+\beta\phi|\phi|^2=0,\quad \beta\neq 0.$$

(a) *Show that the function $\psi:\mathbb{R}\to\mathbf{C}$ defining the traveling wave solution $\phi(x,t)=\psi(x-ct)$ for $c>0$ satisfies a second-order complex differential equation equivalent to a Hamiltonian system in $\mathbb{R}^4$ relative to the noncanonical symplectic form whose matrix is given by*

$$w_c=\begin{bmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&-c\\0&-1&c&0\end{bmatrix}.$$

(b) *Analyze the equilibria of the resulting Hamiltonian system in $\mathbb{R}^4$ and determine their linear stability properties.*

(c) *Let $\psi(s)=e^{ics/2}a(s)$ for a real function $a(s)$ and determine a second order equation for $a(s)$. Show that the resulting equation is Hamiltonian and has heteroclinic orbits for $\beta<0$. Find them.*

(d) *Find 'soliton' solutions for the complex nonlinear Schrödinger equation.*

(Luz V. Vela-Arevalo)

Exercise 6.2 Symplectic group/algebra

Show that if a matrix C satisfies (6.10), then $\exp(sC)$ is a symplectic matrix.

Exercise 6.3 When is a linear transformation canonical?

(a) *Let A be a $[n\times n]$ invertible matrix. Show that the map $\phi:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ given by $(\mathbf{q},\mathbf{p})\mapsto(A\mathbf{q},(A^{-1})^T\mathbf{p})$ is a canonical transformation.*

(b) *If $\mathbf{R}$ is a rotation in $\mathbb{R}^3$, show that the map $(\mathbf{q},\mathbf{p})\mapsto(\mathbf{R}\,\mathbf{q},\mathbf{R}\,\mathbf{p})$ is a canonical transformation.*

(Luz V. Vela-Arevalo)

Exercise 6.4 Determinant of symplectic matrices. *Show that the determinant of a symplectic matrix is $+1$, by going through the following steps:*

(a) *use (6.19) to prove that for eigenvalue pairs each member has the same multiplicity (the same holds for quartet members),*

(b) *prove that the* joint *multiplicity of $\lambda=\pm1$ is even,*

(c) *show that the multiplicities of $\lambda=1$ and $\lambda=-1$ cannot be both odd. (Hint: write*

$$P(\lambda)\,=\,(\lambda-1)^{2m+1}(\lambda+1)^{2l+1}Q(\lambda)$$

and show that $Q(1)=0$).

Exercise 6.5 Cherry's example. *What follows refs. [6.2, 6.3] is mostly a reading exercise, about a Hamiltonian system that is linearly stable but nonlinearly unstable. Consider the Hamiltonian system on $\mathbb{R}^4$ given by*

$$H = \frac{1}{2}(q_1^2 + p_1^2) - (q_2^2 + p_2^2) + \frac{1}{2}p_2(p_1^2 - q_1^2) - q_1 q_2 p_1 .$$

(a) *Show that this system has an equilibrium at the origin, which is linearly stable. (The linearized system consists of two uncoupled oscillators with frequencies in ratios 2:1).*

(b) *Convince yourself that the following is a family of solutions parameterize by a constant τ:*

$$\begin{aligned} q_1 &= -\sqrt{2}\frac{\cos(t-\tau)}{t-\tau}, \quad & q_2 &= \frac{\cos 2(t-\tau)}{t-\tau}, \\ p_1 &= \sqrt{2}\frac{\sin(t-\tau)}{t-\tau}, \quad & p_2 &= \frac{\sin 2(t-\tau)}{t-\tau}. \end{aligned}$$

These solutions clearly blow up in finite time; however they start at $t = 0$ at a distance $\sqrt{3}/\tau$ from the origin, so by choosing τ large, we can find solutions starting arbitrarily close to the origin, yet going to infinity in a finite time, so the origin is nonlinearly unstable.

(Luz V. Vela-Arevalo)