Exercises

Exercise 11.1 A Smale horseshoe. *The Hénon map*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 - ax^2 + y \\ bx \end{bmatrix} \tag{11.13}$$

maps the (x, y) plane into itself - it was constructed by Hénon [3.2] in order to mimic the Poincaré section of once-folding map induced by a flow like the one sketched in figure 10.5. For definitiveness fix the parameters to $a = 6$, $b = -1$.

a) *Draw a rectangle in the (x, y) plane such that its nth iterate by the Hénon map intersects the rectangle 2^n times.*

b) *Construct the inverse of the (11.13).*

c) *Iterate the rectangle back in the time; how many intersections are there between the n forward and m backward iterates of the rectangle?*

d) *Use the above information about the intersections to guess the (x, y) coordinates for the two fixed points, a 2-cycle point, and points on the two distinct 3-cycles from table 10.1. The exact cycle points are computed in exercise 12.10.*

Exercise 11.2 Kneading Danish pastry. *Write down the $(x, y) \to (x, y)$ mapping that implements the baker's map of figure 11.4, together with the inverse mapping. Sketch a few rectangles in symbol square and their forward and backward images. (Hint: the mapping is very much like the tent map (10.6)).*

Exercise 11.3 Kneading Danish without flipping. *The baker's map of figure 11.4 includes a flip - a map of this type is called an orientation reversing once-folding map. Write down the $(x, y) \to (x, y)$ mapping that implements an orientation preserving baker's map (no flip; Jacobian determinant = 1). Sketch and label the first few foldings of the symbol square.*

Exercise 11.4 Fix this manuscript. *Check whether the layers of the baker's map of figure 11.4 are indeed ordered as the branches of the alternating binary tree of figure 10.9. (They might not be - we have not rechecked them). Draw the correct binary trees that order both the future and past itineraries.*

For once-folding maps there are four topologically distinct ways of laying out the stretched and folded image of the starting region,

(a) *orientation preserving: stretch, fold upward, as in figure 11.5*

(b) *orientation preserving: stretch, fold downward, as in figure 13.2*

(c) *orientation reversing: stretch, fold upward, flip, as in figure 11.6*

(d) *orientation reversing: stretch, fold downward, flip, as in figure 11.4,*

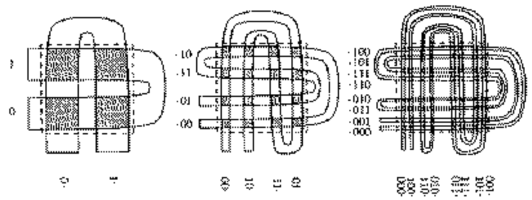


Figure 11.5: A complete Smale horseshoe iterated forwards and backwards, orientation preserving case: function f maps the dashed border square $\mathcal{M}$ into the vertical horseshoe, while the inverse map f^{-1} maps it into the horizontal horseshoe. a) One iteration, b) two iterations, c) three iterations. The non–wandering set is contained within the intersection of the forward and backward iterates (crosshatched). (from K.T. Hansen [1.7])

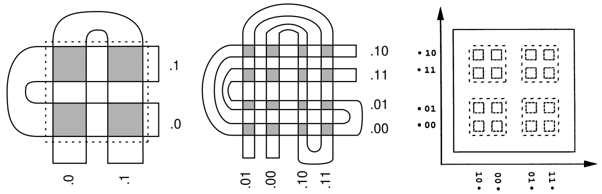


Figure 11.6: An orientation reversing Smale horseshoe map. Function $f = \{$stretch,fold,flip$\}$ maps the dashed border square $\mathcal{M}$ into the vertical horseshoe, while the inverse map f^{-1} maps it into the horizontal horseshoe. a) one iteration, b) two iterations, c) the non–wandering set cover by 16 rectangles, each labeled by the 2 past and the 2 future steps. (from K.T. Hansen [1.7])

with the corresponding four distinct binary-labeled symbol squares. For n-fold "stretch & fold" flows the labeling would be $n$ary. The intersection $\mathcal{M}_0$ for the orientation preserving Smale horseshoe, figure 11.5a, is oriented the same way as $\mathcal{M}$, while $\mathcal{M}_1$ is oriented opposite to $\mathcal{M}$. Brief contemplation of figure 11.4 indicates that the forward iteration strips are ordered relative to each other as the branches of the alternating binary tree in figure 10.9.

Check the labeling for all four cases.

Exercise 11.5 Orientation reversing once-folding map. *By adding a reflection around the vertical axis to the horseshoe map g we get the orientation reversing map $\tilde{g}$ shown in figure 11.6. $\tilde{Q}_0$ and $\tilde{Q}_1$ are oriented as Q_0 and Q_1, so the definition of the future topological coordinate γ is identical to the γ for the orientation preserving horseshoe. The inverse intersections $\tilde{Q}_0^{-1}$ and $\tilde{Q}_1^{-1}$ are oriented so that $\tilde{Q}_0^{-1}$ is opposite to Q, while $\tilde{Q}_1^{-1}$ has the same orientation as Q. Check that the past topological coordinate δ is given by*

$$\begin{aligned} w_{n-1} &= \begin{cases} 1 - w_n & \text{if } s_n = 0 \\ w_n & \text{if } s_n = 1 \end{cases}, \qquad w_0 = s_0 \\ \delta(x) &= 0.w_0 w_{-1} w_{-2} \ldots = \sum_{n=1}^{\infty} w_{1-n}/2^n . \end{aligned} \tag{11.14}$$

Exercise 11.6 Infinite symbolic dynamics. *Let σ be a function that returns zero or one for every infinite binary string: $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}$. Its value is represented by $\sigma(\epsilon_1, \epsilon_2, \ldots)$ where the ϵ_i are either 0 or 1. We will now define an operator $\mathcal{T}$ that acts on observables on the space of binary strings. A function a is an observable if it has bounded variation, that is, if*

$$\|a\| = \sup_{\{\epsilon_i\}} |a(\epsilon_1, \epsilon_2, \ldots)| < \infty .$$

For these functions

$$\begin{aligned} \mathcal{T}a(\epsilon_1, \epsilon_2, \ldots) &= a(0, \epsilon_1, \epsilon_2, \ldots)\sigma(0, \epsilon_1, \epsilon_2, \ldots) \\ &\quad + a(1, \epsilon_1, \epsilon_2, \ldots)\sigma(1, \epsilon_1, \epsilon_2, \ldots) . \end{aligned}$$

(a) *(easy) Consider a finite version T_n of the operator $\mathcal{T}$:*

$$\begin{aligned} &T_n a(\epsilon_1, \epsilon_2, \ldots, \epsilon_{1,n}) = \\ &\quad a(0, \epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1})\sigma(0, \epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1}) + \\ &\quad a(1, \epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1})\sigma(1, \epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1}) . \end{aligned}$$

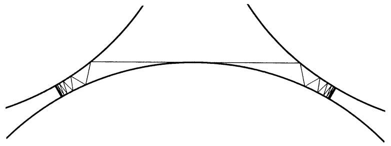
Show that T_n is a $2^n \times 2^n$ matrix. Show that its trace is bounded by a number independent of n.

(b) *(medium) With the operator norm induced by the function norm, show that $\mathcal{T}$ is a bounded operator.*

(c) *(hard) Show that $\mathcal{T}$ is not trace class. (Hint: check if $\mathcal{T}$ is compact "trace class" is defined in appendix J.)*

Exercise 11.7 Time reversability.** *Hamiltonian flows are time reversible. Does that mean that their Markov graphs are symmetric in all node $\to$ node links, their transition matrices are adjacency matrices, symmetric and diagonalizable, and that they have only real eigenvalues?*

Exercise 11.8 3-disk pruning *(Not easy) Show that for 3-disk game of pinball the pruning of orbits starts at $R : a = 2.04821419\ldots$.*



(Kai T. Hansen)

Exercise 11.9 Alphabet {0,1}, prune _1000_, _00100_, _01100_. *This example is motivated by the pruning front description of the symbolic dynamics for the Hénon-type maps.*

step 1. *_1000_ prunes all cycles with a _000_ subsequence with the exception of the fixed point $\overline{0}$; hence we factor out $(1-t_0)$ explicitly, and prune _000_ from the rest. This means that x_0 is an isolated fixed point - no cycle stays in its vicinity for more than 2 iterations. In the notation of sect. 11.5.1, the alphabet is $\{1, 2, 3; \overline{0}\}$, and the remaining pruning rules have to be rewritten in terms of symbols 2=10, 3=100:*

step 2. *alphabet $\{1, 2, 3; \overline{0}\}$, prune _33_, _213_, _313_. This means that the 3-cycle $\overline{3} = \overline{100}$ is pruned and no long cycles stay close enough to it for a single _100_ repeat. As in example 1?!, prohibition of _33_ is implemented by dropping the symbol "3" and extending the alphabet by the allowed blocks 13, 23:*

step 3. *alphabet $\{1, 2, \underline{13}, \underline{23}; \overline{0}\}$, prune $\_2\underline{13}\_$, $\_\underline{23}\,\underline{13}\_$, $\_\underline{13}\,\underline{13}\_$, where $\underline{13} = 13$, $\underline{23} = 23$ are now used as single letters. Pruning of the repetitions $\_\underline{13}\,\underline{13}\_$ (the 4-cycle $\overline{13} = \overline{1100}$ is pruned) yields the*

result: *alphabet $\{1, 2, \underline{23}, \underline{113}; \overline{0}\}$, unrestricted 4-ary dynamics. The other remaining possible blocks $\_\underline{213}\_$, $\_\underline{2313}\_$ are forbidden by the rules of step 3. (continued as exercise 13.20)*