Exercises

Exercise 4.1 **Trace-log of a matrix.** *Prove that*

$$\det\, M = e^{\mathrm{tr}\,\ln M}\,.$$

for an arbitrary nonsingular finite dimensional matrix M, $\det M \neq 0$.

Exercise 4.2 **Stability, diagonal case.** *Verify the relation (4.17)*

$$J^t = e^{t\mathbf{A}} = \mathbf{U}^{-1} e^{t\mathbf{A}_D}\mathbf{U}\,, \quad \textit{where} \quad \mathbf{A}_D = \mathbf{U}\mathbf{A}\mathbf{U}^{-1}\,.$$

Exercise 4.3 **State space volume contraction in Rössler flow.**

(a) *Compute the Rössler flow volume contraction rate at the equilibria.*

(b) *Study numerically the instantaneous* $\partial_i v_i$ *along a typical trajectory on the Rössler attractor; color-code the points on the trajectory by the sign (and perhaps the magnitude) of* $\partial_i v_i$*. If you see regions of local expansion, explain them.*

(c) *Compute numerically the average contraction rate (4.47) along a typical trajectory on the Rössler attractor.*

(d) *(optional) color-code the points on the trajectory by the sign (and perhaps the magnitude) of* $\partial_i v_i - \overline{\partial_i v_i}$.

(e) *Argue on basis of your results that this attractor is of dimension smaller than the state space* $d = 3$.

(f) *(optional) Start some trajectories on the escape side of the outer equilibrium, color-code the points on the trajectory. Is the flow volume contracting?*

Exercise 4.4 **Topology of the Rössler flow.** *(continuation of exercise 3.1)*

(a) *Show that equation* $|\det(A-\lambda\mathbf{1})| = 0$ *for Rössler system in the notation of exercise 2.21 can be written as*

$$\lambda^3 + \lambda^2 c\,(p^{\mp} - \epsilon) + \lambda(p^{\pm}/\epsilon + 1 - c^2\epsilon p^{\mp}) \mp c\sqrt{D} = 0 \qquad (4.58)$$

(b) *Solve (4.58) for eigenvalues* $\lambda^{\pm}$ *for each equilibrium as an expansion in powers of* ϵ*. Derive*

$$\begin{array}{l} \lambda_1^- = -c + \epsilon c/(c^2+1) + o(\epsilon) \\ \lambda_2^- = \epsilon c^3/[2(c^2+1)] + o(\epsilon^2) \\ \theta_2^- = 1 + \epsilon/[2(c^2+1)] + o(\epsilon) \\ \lambda_1^+ = c\epsilon(1-\epsilon) + o(\epsilon^3) \\ \lambda_2^+ = -\epsilon^5 c^2/2 + o(\epsilon^6) \\ \theta_2^+ = \sqrt{1+1/\epsilon}\,(1+o(\epsilon)) \end{array} \qquad (4.59)$$

Compare with exact eigenvalues. What are dynamical implications of the extravagant value of λ_1^-*? (continued as exercise 12.7)*

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Exercise 4.5 Time-ordered exponentials. *Given a time dependent matrix $V(t)$ check that the time-ordered exponential*

$$\mathcal{U}(t) = \mathbf{T} e^{\int_0^t d\tau V(\tau)}$$

may be written as

$$\mathcal{U}(t) = \sum_{m=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m V(t_1) \cdots V(t_m)$$

and verify, by using this representation, that $\mathcal{U}(t)$ satisfies the equation

$$\dot{\mathcal{U}}(t) = V(t)\mathcal{U}(t),$$

with the initial condition $\mathcal{U}(0) = 1$.

Exercise 4.6 A contracting baker's map. *Consider a contracting (or 'dissipative') baker's map, acting on a unit square $[0,1]^2 = [0,1] \times [0,1]$, defined by*

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n/3 \\ 2y_n \end{pmatrix} \quad y_n \le 1/2$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n/3 + 1/2 \\ 2y_n - 1 \end{pmatrix} \quad y_n > 1/2\,.$$

This map shrinks strips by a factor of 1/3 in the x-direction, and then stretches (and folds) them by a factor of 2 in the y-direction.

By how much does the state space volume contract for one iteration of the map?