Exercises

Exercise 31.1 **Who ordered $\sqrt{\pi}$?** *Derive the Gaussian integral*

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2a}} = \sqrt{a}\,, \qquad a>0\,.$$

assuming only that you know to integrate the exponential function e^{-x}. Hint, hint: x^2 is a radius-squared of something. π is related to the area or circumference of something.

Exercise 31.2 **Dirac delta function, Gaussian representation.** *Consider the Gaussian distribution function*

$$\delta_\sigma(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-z^2/2\sigma^2}\,.$$

Show that in $\sigma \to 0$ limit this is the Dirac delta function

$$\int_{\mathcal{M}} dx\, \delta(x) = 1 \text{ if } 0\in\mathcal{M}\,, \text{ zero otherwise}\,.$$

Exercise 31.3 **D-dimensional Gaussian integrals.** *Show that the Gaussian integral in D-dimensions is given by*

$$\frac{1}{(2\pi)^{d/2}}\int d^d\phi e^{-\frac{1}{2}\phi^T\cdot M^{-1}\cdot\phi+\phi\cdot J} \;=\; |\det M|^{\frac{1}{2}}\, e^{\frac{1}{2}J^T\cdot M\cdot J}\,, \qquad (31.53)$$

where M is a real positive definite $[d\times d]$ matrix, that is a matrix with strictly positive eigenvalues. x, J are D-dimensional vectors, and x^T is the transpose of x.

Exercise 31.4 **Stationary phase approximation in higher dimensions.** *All semiclassical approximations are based on saddlepoint evaluations of integrals of type*

$$I = \int d^D x A(x) e^{i\Phi(x)/\hbar} \qquad (31.54)$$

for small values of $\hbar$. Obtain the stationary phase estimate

$$I \approx \sum_n A(x_n) e^{i\Phi(x_n)/\hbar} \frac{(2\pi i\hbar)^{D/2}}{\sqrt{\det \mathbf{D}^2\Phi(x_n)}},$$

where $\mathbf{D}^2\Phi(x_n)$ denotes the second derivative matrix.

Exercise 31.5 **Schrödinger equation in the Madelung form.** *Verify the decomposition of Schrödinger equation into real and imaginary parts, eqs. (31.3) and (31.4).*

Exercise 31.6 **Transport equations.** *Write the wave-function in the asymptotic form*

$$\psi(q,t) = e^{\frac{i}{\hbar}R(x,t)+\frac{i}{\hbar}\varepsilon t}\sum_{n\geq 0}(i\hbar)^n A_n(x,t)\,.$$

Derive the transport equations for the A_n by substituting this into the Schrödinger equation and then collecting terms by orders of $\hbar$. Notice that equation for $\dot{A}_n$ only requires knowledge of A_{n-1} and R.

Exercise 31.7 **Easy examples of the Hamilton's principal function.** *Calculate $R(q,q',t)$ for*

a) a D-dimensional free particle

b) a 3-dimensional particle in constant magnetic field

c) a 1-dimensional harmonic oscillator.

(Continuation: exercise 31.15.)

Exercise 31.8 **1-dimensional harmonic oscillator.** *Take a 1-dimensional harmonic oscillator $U(q)=\frac{1}{2}kq^2$. Take a WKB wave function of form $A(q,t)=a(t)$ and $R(q,t)=r(t)+b(t)q+c(t)q^2$, where $r(t), a(t), b(t)$ and $c(t)$ are time dependent coefficients. Derive ordinary differential equations by using (31.3) and (31.4) and solve them. (Continuation: exercise 31.11.)*

Exercise 31.9 **1-dimensional linear potential.** *Take a 1-dimensional linear potential $U(q)=-Fq$. Take a WKB wave function of form $A(q,t)=a(t)$ and $R(q,t)=r(t)+b(t)q+c(t)q^2$, where $r(t), a(t), b(t)$ and $c(t)$ are time dependent coefficients. Derive and solve the ordinary differential equations from (31.3) and (31.4).*

Exercise 31.10 **D-dimensional quadratic potentials.** *Generalize the above method to general D-dimensional quadratic potentials.*

Exercise 31.11 **Time evolution of R.** *(Continuation of exercise 31.8). Calculate the time evolution of $R(q,0)=a+bq+cq^2$ for a 1-dimensional harmonic oscillator using (31.12) and (31.14).*

Exercise 31.12 **D-dimensional free particle propagator.** *Verify the results in sect. 31.2.2; show explicitly that (31.34), the semiclassical Van Vleck propagator in D dimensions, solves the Schrödinger's equation.*

Exercise 31.13 Propagator, charged particle in constant magnetic field. *Calculate the semiclassical propagator for a charged particle in constant magnetic field in 3 dimensions. Verify that the semiclassical expression coincides with the exact solution.*

Exercise 31.14 1-dimensional harmonic oscillator propagator. *Calculate the semiclassical propagator for a 1-dimensional harmonic oscillator and verify that it is identical to the exact quantum propagator.*

Exercise 31.15 Free particle action. *Calculate the energy dependent action for a free particle, a charged particle in a constant magnetic field and for the harmonic oscillator.*

Exercise 31.16 Zero length orbits. *Derive the classical trace (16.1) rigorously and either add the $t \to 0_+$ zero length contribution to the trace formula, or show that it vanishes. Send us a reprint of* Phys. Rev. Lett. *with the correct derivation.*

Exercise 31.17 Free particle semiclassical Green's functions. *Calculate the semiclassical Green's functions for the systems of exercise 31.15.*