H.1 Deterministic diffusion, zig-zag map

To illustrate the main idea of chapter 13, tracking of a globally diffusing orbit by the associated confined orbit restricted to the fundamental cell, we consider a class of simple 1-d dynamical systems, chains of piecewise linear maps, where all transport coefficients can be evaluated analytically. The translational symmetry (13.10) relates the unbounded dynamics on the real line to the dynamics restricted to a "fundamental cell" - in the present example the unit interval curled up into a circle. An example of such map is the sawtooth map

$$\hat{f}(x) = \begin{cases} \Lambda x & x \in [0, 1/4 + 1/4\Lambda] \\ -\Lambda x + (\Lambda+1)/2 & x \in [1/4 + 1/4\Lambda, 3/4 - 1/4\Lambda] \\ \Lambda x + (1-\Lambda) & x \in [3/4 - 1/4\Lambda, 1] \end{cases} . \tag{H.1}$$

The corresponding circle map $f(x)$ is obtained by modulo the integer part. The elementary cell map $f(x)$ is sketched in fig. H.1. The map has the symmetry property

$$\hat{f}(\hat{x}) \;=\; -\hat{f}(-\hat{x}) , \tag{H.2}$$

so that the dynamics has no drift, and all odd derivatives of the generating function (13.3) with respect to β evaluated at $\beta = 0$ vanish.

The cycle weights are given by

$$t_p = z^{n_p} \frac{e^{\beta \hat{n}_p}}{|\Lambda_p|} . \tag{H.3}$$

The diffusion constant formula for 1-d maps is

$$D = \frac{1}{2} \frac{\left\langle \hat{n}^2 \right\rangle_\zeta}{\langle n \rangle_\zeta} \tag{H.4}$$

where the "mean cycle time" is given by

$$\langle n \rangle_\zeta = z \frac{\partial}{\partial z} \frac{1}{\zeta(0,z)} \bigg|_{z=1} = -\sum{}' (-1)^k \frac{n_{p_1} + \cdots + n_{p_k}}{|\Lambda_{p_1} \cdots \Lambda_{p_k}|} , \tag{H.5}$$

the mean cycle displacement squared by

$$\left\langle \hat{n}^2 \right\rangle_\zeta = \frac{\partial^2}{\partial \beta^2} \frac{1}{\zeta(\beta,1)} \bigg|_{\beta=0} = -\sum{}' (-1)^k \frac{(\hat{n}_{p_1} + \cdots + \hat{n}_{p_k})^2}{|\Lambda_{p_1} \cdots \Lambda_{p_k}|} , \tag{H.6}$$

and the sum is over all distinct non-repeating combinations of prime cycles. Most of results expected in this projects require no more than pencil and paper computations.

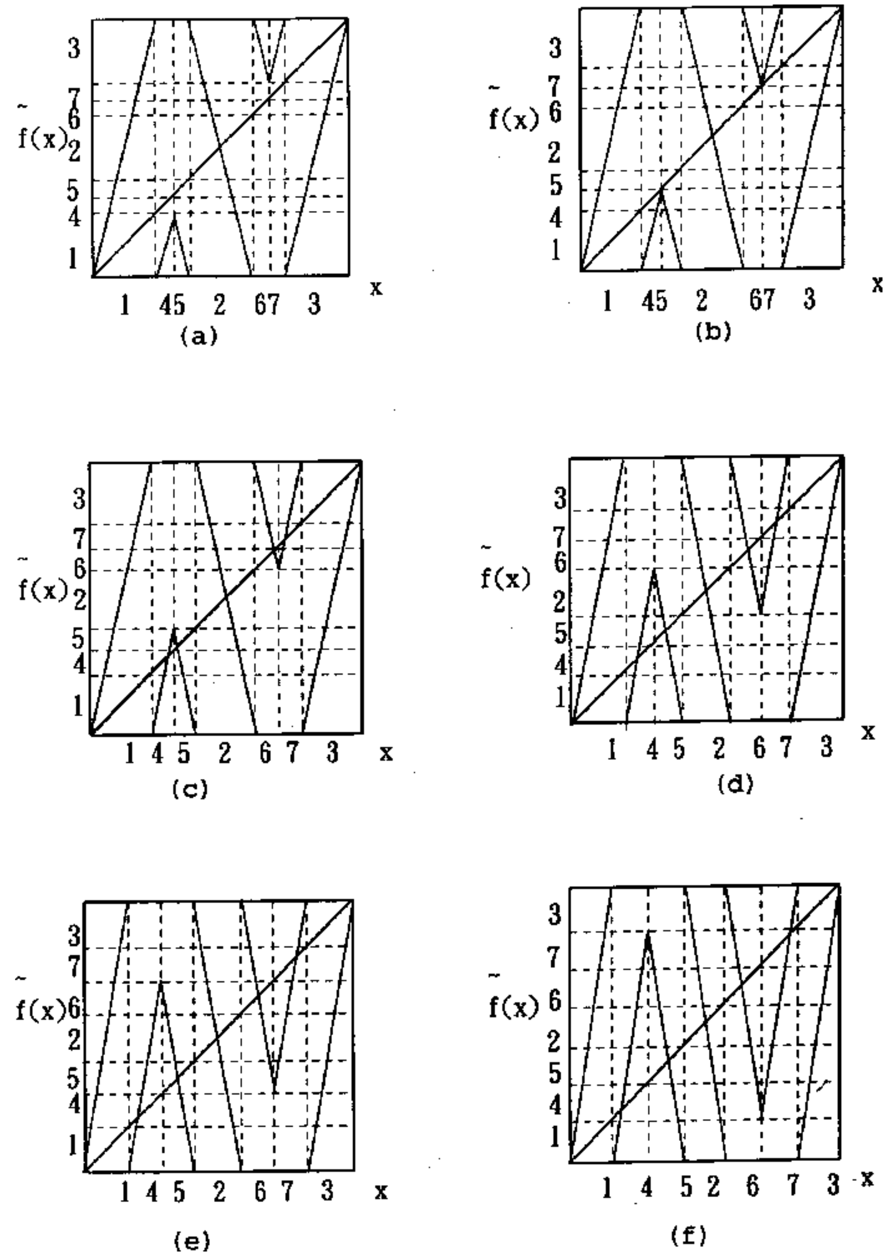


Figure H.1: (a)-(f) The sawtooth map (H.1) for the 6 values of parameter a for which the folding point of the map aligns with the endpoint of one of the 7 intervals and yields a finite Markov partition (from ref. [H.1]). The corresponding Markov graphs are given in fig. H.2.

Implementing the symmetry factorization (13.28) is convenient, but not essential for this project, so if you find sect. 17.1.2 too long a read, skip the symmetrization.

H.1.1 The full shift

Take the map (H.1) and extend it to the real line. As in example of fig. 13.2, denote by a the critical value of the map (the maximum height in the unit cell)

$$a = \hat{f}(\frac{1}{4} + \frac{1}{4\Lambda}) = \frac{\Lambda + 1}{4}\,. \tag{H.7}$$

Describe the symbolic dynamics that you obtain when a is an integer, and derive the formula for the diffusion constant:

$$D \; = \; \frac{(\Lambda^2 - 1)(\Lambda - 3)}{96\Lambda} \qquad \text{for } \Lambda = 4a - 1, \; a \in \mathbb{Z}\,. \tag{H.8}$$

If you are going strong, derive also the fromula for the half-integer $a = (2k+1)/2$, $\Lambda = 4a + 1$ case and email it to DasBuch@nbi.dk. You will need to partition $\mathcal{M}_2$ into the left and right half, $\mathcal{M}_2 = \mathcal{M}_8 \cup \mathcal{M}_9$, as in the derivation of (13.20). exercise 13.1

H.1.2 Subshifts of finite type

We now work out an example when the partition is Markov, although the slope is not an integer number. The key step is that of having a partition where intervals are mapped *onto* unions of intervals. Consider for example the case in which $\Lambda = 4a - 1$, where $1 \le a \le 2$. A first partition is constructed from seven intervals, which we label $\{\mathcal{M}_1, \mathcal{M}_4, \mathcal{M}_5, \mathcal{M}_2, \mathcal{M}_6, \mathcal{M}_7, \mathcal{M}_3\}$, with the alphabet ordered as the intervals are laid out along the unit interval. In general the critical value a will not correspond to an interval border, but now we choose a such that the critical point is mapped onto the right border of $\mathcal{M}_1$, as in fig. H.1(a). The critical value of $f()$ is $f(\frac{\Lambda+1}{4\Lambda}) = a - 1 = (\Lambda - 3)/4$. Equating this with the right border of $\mathcal{M}_1$, $x = 1/\Lambda$, we obtain a quadratic equation with the expanding solution $\Lambda = 4$. We have that $f(\mathcal{M}_4) = f(\mathcal{M}_5) = \mathcal{M}_1$, so the transition matrix (2.2) is given by

$$\phi' = T\phi \; = \; \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_4 \\ \phi_5 \\ \phi_2 \\ \phi_6 \\ \phi_7 \\ \phi_3 \end{pmatrix} \tag{H.9}$$

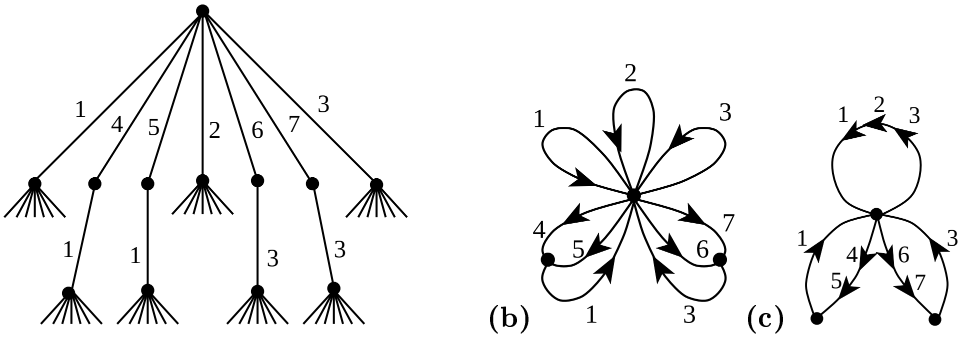
(a)  (b) (c)

Figure H.2: (a) The sawtooth map (H.1) partition tree for fig. H.1(a); while intervals $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ map onto the whole unit interval, $f(\mathcal{M}_1) = f(\mathcal{M}_2) = f(\mathcal{M}_3) = \mathcal{M}$, intervals $\mathcal{M}_4, \mathcal{M}_5$ map onto $\mathcal{M}_1$ only, $f(\mathcal{M}_4) = f(\mathcal{M}_5) = \mathcal{M}_1$, and similarly for intervals $\mathcal{M}_6, \mathcal{M}_7$. An initial point starting out in the interval $\mathcal{M}_1$, $\mathcal{M}_2$ or $\mathcal{M}_3$ can land anywhere on the unit interval, so the subtrees originating from the corresponding nodes on the partition three are similar to the whole tree and can be identified (as, for example, in fig. 2.15), yielding (b) the Markov graph for the Markov partition of fig. H.1(a). (c) the Markov graph in the compact notation of (13.25).

and the dynamics is unrestricted in the alphabet

$$\{1,\, \underline{41},\, \underline{51},\, 2,\, \underline{63},\, \underline{73},\, 3, \} .$$

One could diagonalize (H.9) on the computer, but, as we saw in sect. 2.6, the Markov graph fig. H.2(b) corresponding to fig. H.1(a) offers more insight into the dynamics. The dynamical zeta function

$$\begin{aligned} 1/\zeta &= 1 - (t_1 + t_2 + t_3) - 2(t_{14} + t_{37}) \\ 1/\zeta &= 1 - 3\frac{z}{\Lambda} - 4\cosh\beta\frac{z^2}{\Lambda^2} . \end{aligned} \tag{H.10}$$

follows from the loop expansion (3.10) of sect. 3.3.

The material flow conservation sect. 11.1 and the symmetry factorization (13.28) yield

$$0 = \frac{1}{\zeta(0,1)} = \left(1 + \frac{1}{\Lambda}\right)\left(1 - \frac{4}{\Lambda}\right)$$

which indeed is satisfied by the given value of Λ. Conversely, we can use the desired Markov partition topology to write down the corresponding dynamical zeta function, and use the $1/\zeta(0,1) = 0$ condition to fix Λ. For more complicated transition matrices the factorization (13.28) is very helpful in reducing the order of the polynomial condition that fixes Λ.

fig. H.1	Λ	D
	3	0
(a)	4	$\frac{1}{10}$
(b)	$\sqrt{5}+2$	$\frac{1}{2\sqrt{5}}$
(c)	$\frac{1}{2}(\sqrt{17}+5)$	$\frac{2}{\sqrt{17}}$
(c')	5	$\frac{2}{5}$
(d)	$\frac{1}{2}(\sqrt{33}+5)$	$\frac{1}{8}+\frac{5}{88}\sqrt{33}$
(e)	$2\sqrt{2}+3$	$\frac{1}{2\sqrt{2}}$
(f)	$\frac{1}{2}(\sqrt{33}+7)$	$\frac{1}{4}+\frac{1}{4\sqrt{33}}$
	7	$\frac{2}{7}$

Table H.1: The diffusion constant as function of the slope Λ for the $a = 1, 2$ values of (H.8) and the 6 Markov partitions of fig. H.1

The diffusion constant follows from (13.29) and (H.4)

$$\langle n \rangle_\zeta = -\left(1+\frac{1}{\Lambda}\right)\left(-\frac{4}{\Lambda}\right) , \quad \langle \hat{n}^2 \rangle_\zeta = \frac{4}{\Lambda^2}$$

$$D = \frac{1}{2}\frac{1}{\Lambda+1} = \frac{1}{10}$$

Think up other non-integer values of the parameter for which the symbolic dynamics is given in terms of Markov partitions: in particular consider the cases illustrated in fig. H.1 and determine for what value of the parameter a each of them is realized. Work out the Markov graph, symmetrization factorization and the diffusion constant, and check the material flow conservation for each case. Derive the diffusion constants listed in table H.1. It is not clear why the final answers tend to be so simple. Numerically, the case of fig. H.1(c) appears to yield the maximal diffusion constant. Does it? Is there an argument that it should be so?

The seven cases considered here (see table H.1, fig. H.1 and (H.8)) are the 7 simplest complete Markov partitions, the criterion being that the critical points map onto partition boundary points. This is, for example, what happens for unimodal tent map; if the critical point is preperiodic to an unstable cycle, the grammar is complete. The simplest example is the case in which the tent map critical point is preperiodic to a unimodal map 3-cycle, in which case the grammar is of golden mean type, with _00_ substring prohibited (see fig. 2.15). In case at hand, the "critical" point is the junction of branches 4 and 5 (symmetry automatically takes care of the other critical point, at the junction of branches 6 and 7), and for the cases considered the critical point maps into the endpoint of each of the seven branches.

One can fill out parameter a axis arbitrarily densely with such points - each of the 7 primary intervals can be subdivided into 7 intervals obtained by 2-nd iterate of the map, and for the critical point mapping into any of those in 2 steps the grammar (and the corresponding cycle expansion) is finite, and so on.

H.1.3 Diffusion coefficient, numerically

(optional:)
Attempt a numerical evaluation of

$$D = \frac{1}{2} \lim_{n\to\infty} \frac{1}{n} \left\langle \hat{x}_n^2 \right\rangle . \tag{H.11}$$

Study the convergence by comparing your numerical results to the exact answers derived above. Is it better to use few initial $\hat{x}$ and average for long times, or to use many initial $\hat{x}$ for shorter times? Or should one fit the distribution of $\hat{x}^2$ with a gaussian and get the D this way? Try to plot dependence of D on Λ; perhaps blow up a small region to show that the dependance of D on the parameter Λ is fractal. Compare with figures in refs. [H.1, H.2, H.4, H.5].

H.1.4 D is a nonuniform function of the parameters

(optional:)
The dependence of D on the map parameter Λ is rather unexpected - even though for larger Λ more points are mapped outside the unit cell in one iteration, the diffusion constant does not necessarily grow. An interpretation of this lack of monotonicity would be interesting.

You can also try applying periodic orbit theory to the sawtooth map (H.1) for a random "generic" value of the parameter Λ, for example $\Lambda = 6$. The idea is to bracket this value of Λ by the nearby ones, for which higher and higher iterates of the critical value $a = (\Lambda+1)/4$ fall onto the partition boundaries, compute the exact diffusion constant for each such approximate Markov partition, and study their conergence toward the value of D for $\Lambda = 6$. Judging how difficult such problem is already for a tent map (see sect. 3.6 and appendix B.1), this is too ambitious for a week-long exam.

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