Chapter 4

Local stability

(R. Mainieri and P. Cvitanović)

So far we have concentrated on description of the trajectory of a single initial point. Our next task is to define and determine the size of a *neighborhood* of $x(t)$. We shall do this by assuming that the flow is locally smooth, and describe the local geometry of the neighborhood by studying the flow linearized around $x(t)$. Nearby points aligned along the stable (contracting) directions remain in the neighborhood of the trajectory $x(t) = f^t(x_0)$; the ones to keep an eye on are the points which leave the neighborhood along the unstable directions. As we shall demonstrate in chapter 16, in hyperbolic systems what matters are the expanding directions. The repercussion are far-reaching: As long as the number of unstable directions is finite, the same theory applies to finite-dimensional ODEs, state space volume preserving Hamiltonian flows, and dissipative, volume contracting infinite-dimensional PDEs.

4.1 Flows transport neighborhoods

As a swarm of representative points moves along, it carries along and distorts neighborhoods. The deformation of an infinitesimal neighborhood is best understood by considering a trajectory originating near $x_0 = x(0)$ with an initial infinitesimal displacement $\delta x(0)$, and letting the flow transport the displacement $\delta x(t)$ along the trajectory $x(x_0, t) = f^t(x_0)$.

4.1.1 Instantaneous shear

The system of linear *equations of variations* for the displacement of the infinitesimally close neighbor $x + \delta x$ follows from the flow equations (2.6)

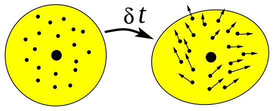

Figure 4.1: A swarm of neighboring points of $x(t)$ is instantaneously sheared by the action of the stability matrix A - a bit hard to draw.

by Taylor expanding to linear order

$$\dot{x}_i + \dot{\delta x}_i = v_i(x + \delta x) \approx v_i(x) + \sum_j \frac{\partial v_i}{\partial x_j}\delta x_j \,.$$

The infinitesimal displacement δx is thus transported along the trajectory $x(x_0, t)$, with time variation given by

$$\frac{d}{dt}\delta x_i(x_0, t) = \sum_j \frac{\partial v_i}{\partial x_j}(x)\bigg|_{x=x(x_0,t)} \delta x_j(x_0, t)\,. \tag{4.1}$$

As both the displacement and the trajectory depend on the initial point x_0 and the time t, we shall often abbreviate the notation to $x(x_0, t) \to x(t) \to x$, $\delta x_i(x_0, t) \to \delta x_i(t) \to \delta x$ in what follows. Taken together, the set of equations

$$\dot{x}_i = v_i(x)\,, \quad \dot{\delta x}_i = \sum_j A_{ij}(x)\delta x_j \tag{4.2}$$

governs the dynamics in the tangent bundle $(x, \delta x) \in \mathbf{T}\mathcal{M}$ obtained by adjoining the d-dimensional tangent space $\delta x \in \mathbf{T}_x\mathcal{M}$ to every point $x \in \mathcal{M}$ in the d-dimensional state space $\mathcal{M} \subset \mathbb{R}^d$. The *stability matrix* (velocity gradients matrix)

$$A_{ij}(x) = \frac{\partial v_i(x)}{\partial x_j} \tag{4.3}$$

describes the instantaneous rate of shearing of the infinitesimal neighborhood of $x(t)$ by the flow, figure 4.1.

Example 4.1 Rössler and Lorenz flows, linearized: *For the Rössler (2.17) and Lorenz (2.12) flows the stability matrices are, respectively*

$$A_{Ross} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x-c \end{pmatrix}, \qquad A_{Lor} = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & x \\ y & x & -b \end{pmatrix}. \tag{4.4}$$

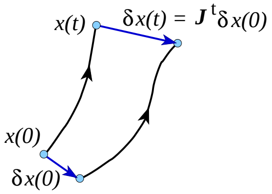

Figure 4.2: The fundamental matrix J^t maps an infinitesimal displacement at x_0 into a displacement rotated and sheared by the linearized flow fundamental matrix $J^t(x_0)$ finite time t later.

4.1.2 Linearized flow

Major combat operations in Iraq have ended.
— President G. W. Bush, May 1, 2003

Taylor expanding a *finite time* flow to linear order,

$$f_i^t(x_0 + \delta x) = f_i^t(x_0) + \sum_j \frac{\partial f_i^t(x_0)}{\partial x_{0j}} \delta x_j + \cdots , \tag{4.5}$$

one finds that the linearized neighborhood is transported by

$$\delta x(t) = J^t(x_0)\delta x(0) , \qquad J_{ij}^t(x_0) = \left. \frac{\partial x_i(t)}{\partial x_j} \right|_{x=x_0} . \tag{4.6}$$

This Jacobian matrix has inherited the name *fundamental solution matrix* or simply *fundamental matrix* from the theory of linear ODEs. It is often denoted Df, but for our needs (we shall have to sort through a plethora of related Jacobian matrices) matrix notation J is more economical. J describes the deformation of an infinitesimal neighborhood at finite time t in the co-moving frame of $x(t)$.

As this is a deformation in the linear approximation, one can think of it as a linear deformation of an infinitesimal sphere enveloping x_0 into an ellipsoid around $x(t)$, described by the eigenvectors and eigenvalues of the fundamental matrix of the linearized flow, figure 4.2. Nearby trajectories separate along the *unstable directions*, approach each other along the *stable directions*, and change their distance along the *marginal directions* at a rate slower than exponential, corresponding to the eigenvalues of the fundamental matrix with magnitude larger than, smaller than, or equal 1. In the literature adjectives *neutral* or *indifferent* are often used instead of 'marginal,' (attracting) stable directions are sometimes called 'asymptotically stable,' and so on.

One of the preferred directions is what one might expect, the direction of the flow itself. To see that, consider two initial points along a trajectory separated by infinitesimal flight time δt: $\delta x(0) = f^{\delta t}(x_0) - x_0 = v(x_0)\delta t$. By the semigroup property of the flow, $f^{t+\delta t} = f^{\delta t + t}$, where

$$f^{\delta t + t}(x_0) = \int_0^{\delta t + t} d\tau \, v(x(\tau)) = \delta t \, v(x(t)) + f^t(x_0) .$$

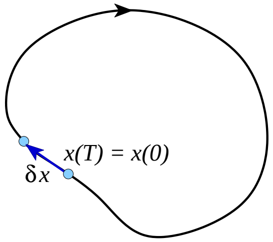

Figure 4.3: For a periodic orbit p, any two points along the cycle are mapped into themselves after one cycle period T, hence $\delta x = v(x_0)\delta t$ is mapped into itself by the cycle fundamental matrix J_p.

Expanding both sides of $f^t(f^{\delta t}(x_0)) = f^{\delta t}(f^t(x_0))$, keeping the leading term in δt, and using the definition of the fundamental matrix (4.6), we observe that $J^t(x_0)$ transports the velocity vector at x_0 to the velocity vector at $x(t)$ at time t:

$$v(x(t)) = J^t(x_0) \, v(x_0) . \tag{4.7}$$

In nomenclature of page 65, the fundamental matrix maps the initial, Lagrangian coordinate frame into the current, Eulerian coordinate frame.

The velocity at point $x(t)$ in general does not point in the same direction as the velocity at point x_0, so this is not an eigenvalue condition for J^t; the fundamental matrix computed for an arbitrary segment of an arbitrary trajectory has no invariant meaning.

As the eigenvalues of finite time J^t have invariant meaning only for periodic orbits, we postpone their interpretation to chapter 5. However, already at this stage we see that if the orbit is periodic, $x(T_p) = x(0)$, at any point along cycle p the velocity v is an eigenvector of the fundamental matrix $J_p = J^{T_p}$ with a unit eigenvalue,

$$J_p(x) \, v(x) = v(x) , \qquad x \in p . \tag{4.8}$$

Two successive points along the cycle separated by $\delta x(0)$ have the same separation after a completed period $\delta x(T_p) = \delta x(0)$, see figure 4.3, hence eigenvalue 1.

As we started by assuming that we know the equations of motion, from (4.3) we also know stability matrix A, the instantaneous rate of shear of an infinitesimal neighborhood $\delta x_i(t)$ of the trajectory $x(t)$. What we do not know is the finite time deformation (4.6).

Our next task is to relate the stability matrix A to fundamental matrix J^t. On the level of differential equations the relation follows by taking the time derivative of (4.6) and replacing $\dot{\delta x}$ by (4.2)

$$\dot{\delta x}(t) = \dot{J}^t \, \delta x(0) = A \delta x(t) = A J^t \, \delta x(0) .$$

Hence the d^2 matrix elements of fundamental matrix satisfy the linearized equation (4.1)

$$\frac{d}{dt} J^t(x) = A(x)\, J^t(x)\,, \quad \text{initial condition} \quad J^0(x) = \mathbf{1}\,. \tag{4.9}$$

Given a numerical routine for integrating the equations of motion, evaluation of the fundamental matrix requires minimal additional programming effort; one simply extends the d-dimensional integration routine and integrates concurrently with $f^t(x)$ the d^2 elements of $J^t(x)$.

The qualifier 'simply' is perhaps too glib. Integration will work for short finite times, but for exponentially unstable flows one quickly runs into numerical over- and/or underflow problems, so further thought will have to go into implementation this calculation.

So now we know how to compute fundamental matrix J^t given the stability matrix A, at least when the d^2 extra equations are not too expensive to compute. Mission accomplished.

fast track:
chapter 6, p. 99

And yet... there are mopping up operations left to do. We persist until we derive the integral formula (4.43) for the fundamental matrix, an analogue of the finite-time "Green function" or "path integral" solutions of other linear problems.

We are interested in smooth, differentiable flows. If a flow is smooth, in a sufficiently small neighborhood it is essentially linear. Hence the next section, which might seem an embarrassment (what is a section on *linear* flows doing in a book on *non*linear dynamics?), offers a firm stepping stone on the way to understanding nonlinear flows. If you know your eigenvalues and eigenvectors, you may prefer to fast forward here.

fast track:
sect. 4.3, p. 73

4.2 Linear flows

> Diagonalizing the matrix: that's the key to the whole thing.
> — Governor Arnold Schwarzenegger

Linear fields are the simplest vector fields, described by linear differential equations which can be solved explicitly, with solutions that are good for all times. The state space for linear differential equations is $\mathcal{M} = \mathbb{R}^d$, and the

equations of motion (2.6) are written in terms of a vector x and a constant stability matrix A as

$$\dot{x} = v(x) = Ax\,. \tag{4.10}$$

Solving this equation means finding the state space trajectory

$$x(t) = (x_1(t), x_2(t), \ldots, x_d(t))$$

passing through the point x_0. If $x(t)$ is a solution with $x(0) = x_0$ and $y(t)$ another solution with $y(0) = y_0$, then the linear combination $ax(t) + by(t)$ with $a, b \in \mathbb{R}$ is also a solution, but now starting at the point $ax_0 + by_0$. At any instant in time, the space of solutions is a d-dimensional vector space, which means that one can find a basis of d linearly independent solutions.

How do we solve the linear differential equation (4.10)? If instead of a matrix equation we have a scalar one, $\dot{x} = \lambda x$, the solution is

$$x(t) = e^{t\lambda} x_0\,. \tag{4.11}$$

In order to solve the d-dimensional matrix case, it is helpful to rederive the solution (4.11) by studying what happens for a short time step δt. If at time $t = 0$ the position is $x(0)$, then

$$\frac{x(\delta t) - x(0)}{\delta t} = \lambda x(0)\,, \tag{4.12}$$

which we iterate m times to obtain Euler's formula for compounding interest

$$x(t) \approx \left(1 + \frac{t}{m}\lambda\right)^m x(0)\,. \tag{4.13}$$

The term in parentheses acts on the initial condition $x(0)$ and evolves it to $x(t)$ by taking m small time steps $\delta t = t/m$. As $m \to \infty$, the term in parentheses converges to $e^{t\lambda}$. Consider now the matrix version of equation (4.12):

$$\frac{x(\delta t) - x(0)}{\delta t} = Ax(0)\,. \tag{4.14}$$

A representative point x is now a vector in $\mathbb{R}^d$ acted on by the matrix A, as in (4.10). Denoting by $\mathbf{1}$ the identity matrix, and repeating the steps (4.12) and (4.13) we obtain Euler's formula for the exponential of a matrix:

$$x(t) = J^t x(0)\,, \qquad J^t = e^{tA} = \lim_{m\to\infty} \left(\mathbf{1} + \frac{t}{m}A\right)^m\,. \tag{4.15}$$

We will find this definition the exponential of a matrix helpful in the general case, where the matrix $A = A(x(t))$ varies along a trajectory.

How do we compute the exponential (4.15)?

in depth:
appendix J.2, p. 807

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Example 4.2 Fundamental matrix eigenvalues, diagonalizable case: *Should we be so lucky that $A = A_D$ happens to be a diagonal matrix with eigenvalues $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)})$, the exponential is simply*

$$J^t = e^{tA_D} = \begin{pmatrix} e^{t\lambda^{(1)}} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & e^{t\lambda^{(d)}} \end{pmatrix} . \tag{4.16}$$

Next, suppose that A is diagonalizable and that U is a nonsingular matrix that brings it to a diagonal form $A_D = U^{-1}AU$. Then J can also be brought to a diagonal form (insert factors $\mathbf{1} = UU^{-1}$ between the terms of the product (4.15)):

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$$J^t = e^{tA} = Ue^{tA_D}U^{-1} . \tag{4.17}$$

The action of both A and J is very simple; the axes of orthogonal coordinate system where A is diagonal are also the eigen-directions of both A and J^t, and under the flow the neighborhood is deformed by a multiplication by an eigenvalue factor for each coordinate axis.

In general J^t is neither diagonal, nor diagonalizable, nor constant along the trajectory. As any matrix, J^t can also be expressed in the singular value decomposition form

$$J = UDV^T$$

where D is diagonal, and U, V are orthogonal matrices. The diagonal elements σ_1, σ_2, …, σ_d of D are called the *singular values* of J, namely the square root of the eigenvalues of $J^\dagger J$, which is a Hermitian, positive semi-definite matrix (and thus admits only real, non-negative eigenvalues). From a geometric point of view, when all singular values are non-zero, J maps the unit sphere into an ellipsoid: the singular values are then the lengths of the semiaxes of this ellipsoid.

sect. 5.1.2

We recapitulate the basic facts of linear algebra in appendix B. A 2–d example serves well to highlight the most important types of linear flows:

Example 4.3 Linear stability of 2–d flows: *For a 2–d flow the eigenvalues $\lambda^{(1)}, \lambda^{(2)}$ of A are either real, leading to a linear motion along their eigenvectors, $x_j(t) = x_j(0)\exp(t\lambda^{(j)})$, or a form a complex conjugate pair $\lambda^{(1)} = \mu + i\omega\,, \lambda^{(2)} = \mu - i\omega\,,$ leading to a circular or spiral motion in the $[x_1, x_2]$ plane.*

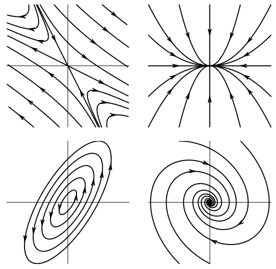

Figure 4.4: Streamlines for several typical 2-dimensional flows: saddle (hyperbolic), in node (attracting), center (elliptic), in spiral.

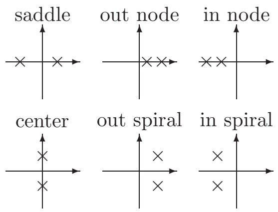

Figure 4.5: Qualitatively distinct types of exponents of a $[2{\times}2]$ fundamental matrix.

These two possibilities are refined further into sub-cases depending on the signs of the real part. In the case $\lambda^{(1)} > 0$, $\lambda^{(2)} < 0$, x_1 grows exponentially with time, and x_2 contracts exponentially. This behavior, called a saddle, *is sketched in figure 4.4, as are the remaining possibilities: in/out nodes, inward/outward spirals, and the center. The magnitude of out-spiral $|x(t)|$ diverges exponentially when $\mu > 0$, and contracts into $(0, 0)$ when the $\mu < 0$, whereas the phase velocity ω controls its oscillations.*

If eigenvalues $\lambda^{(1)} = \lambda^{(2)} = \lambda$ are degenerate, the matrix might have two linearly independent eigenvectors, or only one eigenvector. We distinguish two cases: (a) A can be brought to diagonal form. (b) A can be brought to Jordan form, *which (in dimension 2 or higher) has zeros everywhere except for the repeating eigenvalues on the diagonal, and some 1's directly above it. For every such Jordan $[d_\alpha \times d_\alpha]$ block there is only one eigenvector per block.*

We sketch the full set of possibilities in figures 4.4 and 4.5, and work out in detail the most important cases in appendix B, example B.2.

4.2.1 Eigenvalues, multipliers - a notational interlude

Throughout this text the symbol Λ_k will always denote the kth *eigenvalue* (in literature sometimes referred to as the *multiplier*) of the finite time fundamental matrix J^t. Symbol $\lambda^{(k)}$ will be reserved for the kth *characteristic exponent* or *characteristic value*, with real part $\mu^{(k)}$ and phase $\omega^{(k)}$:

$$\Lambda_k = e^{t\lambda^{(k)}} = e^{t(\mu^{(k)} + i\omega^{(k)})} . \tag{4.18}$$

$J^t(x_0)$ depends on the initial point x_0 and the elapsed time t. For notational brevity we tend to omit this dependence, but in general

$$\Lambda = \Lambda_k = \Lambda_k(x_0, t)\,,\ \lambda = \lambda^{(k)}(x_0, t)\,,\ \omega = \omega^{(k)}(x_0, t)\,, \cdots \text{ etc.}\,,$$

depend on both the trajectory traversed and the choice of coordinates.

However, as we shall see in sect. 5.2, if the stability matrix A or the fundamental matrix J is computed on a flow-invariant set $\mathcal{M}_p$, such as an equilibrium q or a periodic orbit p of period T_p,

$$A_q = A(x_q)\,, \qquad J_p(x) = J^{T_p}(x)\,, \quad x \in \mathcal{M}_p\,, \tag{4.19}$$

(x is any point on the cycle) its eigenvalues

$$\lambda_q^{(k)} = \lambda^{(k)}(x_q)\,,\ \Lambda_{p,k} = \Lambda_k(x, T_p)$$

are flow-invariant, independent of the choice of coordinates and the initial point in the cycle p, so we label them by their q or p label.

We number eigenvalues Λ_k in order of decreasing magnitude

$$|\Lambda_1| \geq |\Lambda_2| \geq \ldots \geq |\Lambda_d|\,. \tag{4.20}$$

Since $|\Lambda_j| = e^{t\mu^{(j)}}$, this is the same as labeling by

$$\mu^{(1)} \geq \mu^{(2)} \geq \ldots \geq \mu^{(d)}\,. \tag{4.21}$$

In dynamics the expanding directions, $|\Lambda_e| > 1$, have to be taken care of first, while the contracting directions $|\Lambda_c| < 1$ tend to take care of themselves, hence the ordering by decreasing magnitude is the natural one.

4.2.2 Yes, but how do you really do it?

Economical description of neighborhoods of equilibria and periodic orbits is afforded by projection operators

$$\mathbf{P}_i = \prod_{j \neq i} \frac{\mathbf{M} - \lambda^{(j)}\mathbf{1}}{\lambda^{(i)} - \lambda^{(j)}}\,, \tag{4.22}$$

where matrix $\mathbf{M}$ is typically either equilibrium stability matrix A, or periodic orbit fundamental matrix $\hat{J}$ restricted to a Poincaré section, as in (4.55).

While usually not phrased in language of projection operators, the requisite linear algebra is standard, and relegated here to appendix B.

Once the distinct non-zero eigenvalues $\{\lambda^{(i)}\}$ are computed, projection operators are polynomials in $\mathbf{M}$ which need no further diagonalizations or orthogonalizations. For each distinct eigenvalue $\lambda^{(i)}$ of $\mathbf{M}$, the colums/rows of $\mathbf{P}_i$

$$(\mathbf{M} - \lambda^{(j)}\mathbf{1})\mathbf{P}_j = \mathbf{P}_j(\mathbf{M} - \lambda^{(j)}\mathbf{1}) = 0\,, \tag{4.23}$$

are the right/left eigenvectors $\mathbf{e}^{(k)}$, $\mathbf{e}_{(k)}$ of $\mathbf{M}$ which (provided $\mathbf{M}$ is not of Jordan type) span the corresponding linearized subspace, and are a convenient starting seed for tracing out the global unstable/stable manifolds.

Matrices $\mathbf{P}_i$ are *orthogonal* and *complete*:

$$\mathbf{P}_i\mathbf{P}_j = \delta_{ij}\mathbf{P}_j\,, \quad (\text{no sum on } j)\,, \qquad \sum_{i=1}^{r} \mathbf{P}_i = \mathbf{1}\,. \tag{4.24}$$

with the dimension of the ith subspace given by $d_i = \mathrm{tr}\,\mathbf{P}_i$. Completeness relation substituted into $\mathbf{M} = \mathbf{M}\,\mathbf{1}$ yields

$$\mathbf{M} = \lambda^{(1)}\mathbf{P}_1 + \lambda^{(2)}\mathbf{P}_2 + \cdots + \lambda^{(r)}\mathbf{P}_r\,. \tag{4.25}$$

As any matrix function $f(\mathbf{M})$ takes the scalar value $f(\lambda^{(i)})$ on the $\mathbf{P}_i$ subspace, $f(\mathbf{M})\mathbf{P}_i = f(\lambda^{(i)})\mathbf{P}_i$, it is easily evaluated through its *spectral decomposition*

$$f(\mathbf{M}) = \sum_i f(\lambda^{(i)})\mathbf{P}_i\,. \tag{4.26}$$

As $\mathbf{M}$ has only real entries, it will in general have either real eigenvalues (over-damped oscillator, for example), or complex conjugate pairs of eigenvalues (under-damped oscillator, for example). That is not surprising, but also the corresponding eigenvectors can be either real or complex. All coordinates used in defining the flow are real numbers, so what is the meaning of a *complex* eigenvector?

If two eigenvalues form a complex conjugate pair, $\{\lambda^{(k)}, \lambda^{(k+1)}\} = \{\mu + i\omega, \mu - i\omega\}$, they are in a sense degenerate: while a real $\lambda^{(k)}$ characterizes a motion along a line, a complex $\lambda^{(k)}$ characterizes a spiralling motion in a plane. We determine this plane by replacing the corresponding complex eigenvectors by their real and imaginary parts, $\{\mathbf{e}^{(k)}, \mathbf{e}^{(k+1)}\} \rightarrow \{\mathrm{Re}\,\mathbf{e}^{(k)}, \mathrm{Im}\,\mathbf{e}^{(k)}\}$, or, in terms of projection operators:

$$\mathbf{P}_k = \frac{1}{2}(\mathbf{R} + i\mathbf{Q})\,, \qquad \mathbf{P}_{k+1} = \mathbf{P}_k^*\,,$$

where $\mathbf{R} = \mathbf{P}_k + \mathbf{P}_{k+1}$ is the subspace decomposed by the kth complex eigenvalue pair, and $\mathbf{Q} = (\mathbf{P}_k - \mathbf{P}_{k+1})/i$, both matrices with real elements. Substitution

$$\begin{pmatrix} \mathbf{P}_k \\ \mathbf{P}_{k+1} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} \mathbf{R} \\ \mathbf{Q} \end{pmatrix},$$

brings the $\lambda^{(k)}\mathbf{P}_k + \lambda^{(k+1)}\mathbf{P}_{k+1}$ complex eigenvalue pair in the spectral decomposition (4.25) into the real form,

$$\begin{pmatrix} \mathbf{P}_k & \mathbf{P}_{k+1} \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix}\begin{pmatrix} \mathbf{P}_k \\ \mathbf{P}_{k+1} \end{pmatrix} = \begin{pmatrix} \mathbf{R} & \mathbf{Q} \end{pmatrix}\begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}\begin{pmatrix} \mathbf{R} \\ \mathbf{Q} \end{pmatrix}, \quad (4.27)$$

where we have dropped the superscript $^{(k)}$ for notational brevity.

To summarize, spectrally decomposed matrix $\mathbf{M}$ (4.25) acts along lines on subspaces corresponding to real eigenvalues, and as a $[2\times 2]$ rotation in a plane on subspaces corresponding to complex eigenvalue pairs.

Now that we have some feeling for the qualitative behavior of eigenvectors and eigenvalues of linear flows, we are ready to return to the nonlinear case.

4.3 Stability of flows

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How do you determine the eigenvalues of the finite time local deformation J^t for a general nonlinear smooth flow? The fundamental matrix is computed by integrating the equations of variations (4.2)

$$x(t) = f^t(x_0), \quad \delta x(x_0, t) = J^t(x_0)\delta x(x_0, 0). \quad (4.28)$$

The equations are linear, so we should be able to integrate them–but in order to make sense of the answer, we derive it step by step.

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4.3.1 Stability of equilibria

For a start, consider the case where x is an equilibrium point (2.8). Expanding around the equilibrium point x_q, using the fact that the stability matrix $A = A(x_q)$ in (4.2) is constant, and integrating,

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$$f^t(x) = x_q + e^{At}(x - x_q) + \cdots, \quad (4.29)$$

we verify that the simple formula (4.15) applies also to the fundamental matrix of an equilibrium point,

$$J^t(x_q) = e^{A_q t}, \qquad A_q = A(x_q). \quad (4.30)$$

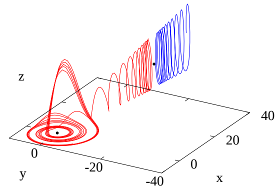

Figure 4.6: Two trajectories of the Rössler flow initiated in the neighborhood of the '+' or 'outer' equilibrium point (2.18). (R. Paškauskas)

Example 4.4 In-out spirals. *Consider a 2-d equilibrium whose stability eigenvalues $\{\lambda^{(1)}, \lambda^{(2)}\} = \{\mu + i\omega, \mu - i\omega\}$ form a complex conjugate pair. The corresponding complex eigenvectors can be replaced by their real and imaginary parts, $\{\mathbf{e}^{(1)}, \mathbf{e}^{(2)}\} \rightarrow \{\mathrm{Re}\,\mathbf{e}^{(k)}, \mathrm{Im}\,\mathbf{e}^{(k)}\}$. The 2-d real representation (4.27),*

$$\begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} = \mu\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \omega\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

consists of the identity and the generator of $SO(2)$ rotations. Trajectories $\mathbf{x}(t) = J^t\,\mathbf{x}(0)$, where

$$J^t = e^{A_q t} = e^{t\mu}\begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}, \quad (4.31)$$

spiral in/out around $(x, y) = (0, 0)$, see figure 4.4, with the rotation period T, and contraction/expansion radially by the multiplier Λ_{radial}, and by the multiplier Λ_j along the $\mathbf{e}^{(j)}$ eigendirection per a turn of the spiral:

$$T = 2\pi/\omega, \qquad \Lambda_{radial} = e^{T\mu}, \qquad \Lambda_j = e^{T\mu^{(j)}}. \quad (4.32)$$

We learn that the typical turnover time scale in the neighborhood of the equilibrium $(x, y) = (0, 0)$ is of order $\approx T$ (and not, let us say, $1000\,T$, or $10^{-2}T$). Λ_j multipliers give us estimates of strange-set thickness.

Example 4.5 Stability of equilibria of the Rössler flow. *The Rösler system (2.17) has two equilibrium points (2.18), the inner equilibrium (x_-, y_-, z_-), and the outer equilibrium point (x^+, y^+, z^+). Together with their exponents (eigenvalues of the stability matrix) the two equilibria now yield quite detailed information about the flow. Figure 4.6 shows two trajectories which start in the neighborhood of the '+' equilibrium point. Trajectories to the right of the outer equilibrium point '+' escape, and those to the left spiral toward the inner equilibrium point '−', where they seem to wander chaotically for all times. The stable manifold of outer equilibrium point thus serves as a attraction basin boundary. Consider now the eigenvalues of the two equilibria*

$$\begin{array}{ll} (\mu_-^{(1)}, \mu_-^{(2)} \pm i\,\omega_-^{(2)}) = & (\,-5.686, \qquad 0.0970 \pm i\,0.9951\,) \\ (\mu_+^{(1)}, \mu_+^{(2)} \pm i\,\omega_+^{(2)}) = & (\,0.1929, \quad -4.596\times 10^{-6} \pm i\,5.428\,) \end{array} \quad (4.33)$$

Outer equilibrium: *The $\mu_+^{(2)} \pm i\,\omega_+^{(2)}$ complex eigenvalue pair implies that that neighborhood of the outer equilibrium point rotates with angular period $T_+ \approx \left|2\pi/\omega_+^{(2)}\right| = 1.1575$. The multiplier by which a trajectory that starts near the '+' equilibrium point contracts in the stable manifold plane is the excrutiatingly slow $\Lambda_2^+ \approx \exp(\mu_+^{(2)}T_+) = 0.9999947$ per rotation. For each period the point of the stable manifold moves away along the unstable eigen-direction by factor $\Lambda_1^+ \approx \exp(\mu_+^{(1)}T_+) = 1.2497$. Hence the* *slow spiraling on both sides of the '+' equilibrium point.*

Inner equilibrium: *The* $\mu_-^{(2)} \pm i\,\omega_-^{(2)}$ *complex eigenvalue pair tells us that neighborhood of the '−' equilibrium point rotates with angular period* $T_- \approx \left|2\pi/\omega_-^{(2)}\right| = 6.313$, *slightly faster than the harmonic oscillator estimate in (2.14). The multiplier by which a trajectory that starts near the '−' equilibrium point spirals away per one rotation is* $\Lambda_{radial} \approx \exp(\mu_-^{(2)} T_-) = 1.84$. *The* $\mu_-^{(1)}$ *eigenvalue is essentially the* z *expansion correcting parameter* c *introduced in (2.16). For each Poincaré section return, the trajectory is contracted into the stable manifold by the amazing factor of* $\Lambda_1 \approx \exp(\mu_-^{(1)} T_-) = 10^{-15.6}$ *(!).*

Suppose you start with a 1 mm *interval pointing in the* Λ_1 *eigen-direction. After one Poincaré return the interval is of order of* 10^{-4} fermi, *the furthest we will get into subnuclear structure in this book. Of course, from the mathematical point of view, the flow is reversible, and the Poincaré return map is invertible.* *(Rytis Paškauskas)*

Example 4.6 Stability of Lorenz flow equilibria: *(Continued from example 3.4.) A glance at figure 3.4 suggests that the flow is organized by its 3 equilibria, so lets have a closer look at their stable/unstable manifolds.*

Lorenz flow is volume contracting (4.47),

$$\partial_i v_i = \sum_{i=1}^{3} \lambda^{(i)}(x,t) = -\sigma - b - 1\,, \tag{4.34}$$

at a constant, coordinate- and ρ*-independent rate, set by Lorenz to* $\partial_i v_i = -13.66$.

The symmetry of Lorenz flow leads to a block-diagonal form for the EQ_0 *equilibrium stability matrix, explicit in (4.4) evaluated at* $x_{EQ0} = (0,0,0)$. *The* z*-axis is an eigenvector with a contracting eigenvalue* $\lambda^{(2)} = -b$. *From (4.34) it follows that all* $[x,y]$ *areas shrink at rate* $-\sigma - b$. *Indeed, the* $[x,y]$ *submatrix*

$$A^- = \begin{pmatrix} -\sigma & \sigma \\ \rho & -1 \end{pmatrix} \tag{4.35}$$

has a real expanding/contracting eigenvalue pair $\lambda^{(1,3)} = -(\sigma+1)/2 \pm \sqrt{(\sigma-1)^2/4 + \rho\sigma}$, *with the right eigenvectors* $\mathbf{e}^{(1)}$, $\mathbf{e}^{(3)}$ *in the* $[x,y]$ *plane, given by (either) column of the projection operator*

$$\mathbf{P}_i = \frac{A^- - \lambda^{(j)}\mathbf{1}}{\lambda^{(i)} - \lambda^{(j)}} = \frac{1}{\lambda^{(i)} - \lambda^{(j)}} \begin{pmatrix} -\sigma - \lambda^{(j)} & \sigma \\ \rho & -1 - \lambda^{(j)} \end{pmatrix}, \qquad i \neq j \in \{1,3\}\,. \tag{4.36}$$

$EQ_{1,2}$ *equilibria have no symmetry, so their eigenvalues are given by the roots of a cubic equation, the secular determinant* $\det(A - \lambda\mathbf{1}) = 0$*:*

$$\lambda^3 + \lambda^2(\sigma + b + 1) + \lambda b(\sigma + \rho) + 2\sigma b(\rho - 1) = 0\,. \tag{4.37}$$

For $\rho > 24.74$, $EQ_{1,2}$ *have one stable real eigenvalue and one unstable complex conjugate pair, leading to a spiral-out instability and the strange attractor depicted in figure 2.4.*

As all numerical plots of the Lorenz flow are here carried out for the Lorenz parameter choice $\sigma = 10, b = 8/3, \rho = 28$, *we note the values of these eigenvalues for future reference,*

$$\begin{array}{lcl} EQ_0: \ (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) & = & (\,11.83\,,\ -2.666,\ \ -22.83\,) \\ EQ_1: \ (\mu^{(1)} \pm i\,\omega^{(1)}, \lambda^{(3)}) & = & (\,0.094 \pm i\,10.19,\ \ -13.85\,)\,, \end{array} \tag{4.38}$$

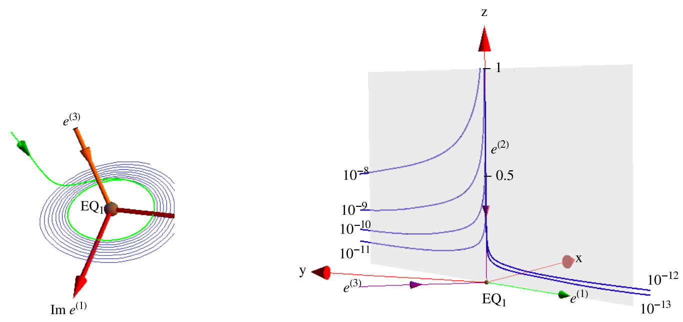

Figure 4.7: (a) A perspective view of the linearized Lorenz flow near EQ_1 equilibrium, see figure 3.4 (a). The unstable eigenplane of EQ_1 is spanned by $\mathrm{Re}\,\mathbf{e}^{(1)}$ and $\mathrm{Im}\,\mathbf{e}^{(1)}$. The stable eigenvector $\mathbf{e}^{(3)}$. (b) Lorenz flow near the EQ_0 equilibrium: unstable eigenvector $\mathbf{e}^{(1)}$, stable eigenvectors $\mathbf{e}^{(2)}$, $\mathbf{e}^{(3)}$. Trajectories initiated at distances $10^{-8} \cdots 10^{-12}$, 10^{-13} away from the z-axis exit finite distance from EQ_0 along the $(\mathbf{e}^{(1)}, \mathbf{e}^{(2)})$ eigenvectors plane. Due to the strong $\lambda^{(1)}$ expansion, the EQ_0 equilibrium is, for all practical purposes, unreachable, and the $EQ_1 \to EQ_0$ heteroclinic connection never observed in simulations such as figure 2.4. (E. Siminos; continued in figure 10.7.)

as well as the rotation period $T_{EQ1} = 2\pi/\omega^{(1)}$ *about* EQ_1, *and the associated expansion/contractio multipliers* $\Lambda^{(i)} = \exp(\mu^{(j)} T_{EQ1})$ *per a spiral-out turn:*

$$T_{EQ1} = 0.6163\,, \qquad (\Lambda^{(1)}, \Lambda^{(3)}) = (\,1.060\,, 1.957 \times 10^{-4}\,)\,. \tag{4.39}$$

We learn that the typical turnover time scale in this problem is of order $T \approx T_{EQ1} \approx 1$ *(and not, let us say, 1000, or* 10^{-2}*). Combined with the contraction rate (4.34), this tells us that the Lorenz flow strongly contracts state space volumes, by factor of* $\approx 10^{-4}$ *per mean turnover time.*

In the EQ_1 *neighborhood the unstable manifold trajectories slowly spiral out, with very small radial per-turn expansion multiplier* $\Lambda^{(1)} \simeq 1.06$, *and very strong contraction multiplier* $\Lambda^{(3)} \simeq 10^{-4}$ *onto the unstable manifold, figure 4.7 (a). This contraction confines, for all practical purposes, the Lorenz attractor to a 2-dimensional surface evident in the section figure 3.4.*

In the $x_{EQ0} = (0,0,0)$ *equilibrium neighborhood the extremely strong* $\lambda^{(3)} \simeq -23$ *contraction along the* $\mathbf{e}^{(3)}$ *direction confines the hyperbolic dynamics near* EQ_0 *to the plane spanned by the unstable eigenvector* $\mathbf{e}^{(1)}$, *with* $\lambda^{(1)} \simeq 12$, *and the slowest contraction rate eigenvector* $\mathbf{e}^{(2)}$ *along the* z*-axis, with* $\lambda^{(2)} \simeq -3$. *In this plane the strong expansion along* $\mathbf{e}^{(1)}$ *overwhelms the slow* $\lambda^{(2)} \simeq -3$ *contraction down the* z*-axis, making it extremely unlikely for a random trajectory to approach* EQ_0, *figure 4.7 (b). Thus linearization suffices to describe analytically the singular dip in the Poincaré sections of figure 3.4, and the empirical scarcity of trajectories close to* EQ_0. *(Continued in example 4.7.)*

(Evangelos Siminos and Jonathan Halcrow)

Example 4.7 Lorenz flow: Global portrait *(Continued from example 4.6.) As the* EQ_1 *unstable manifold spirals out, the strip that starts out in the section above* EQ_1 *in figure 3.4 cuts across the* z*-axis invariant subspace. This strip necessarily contains* *a heteroclinic orbit that hits the z-axis head on, and in infinite time (but exponentially fast) descends all the way to EQ_0.*

*How? As in the neighborhood of the EQ_0 equilibrium the dynamics is linear (see figure **??** (a)), there is no need to integrate numerically the final segment of the heteroclinic connection - it is sufficient to bring a trajectory a small distance away from EQ_0, continue analytically to a small distance beyond EQ_0, then resume the numerical integration.*

What happens next? Trajectories to the left of z-axis shoot off along the $\mathbf{e}^{(1)}$ direction, and those to the right along $-\mathbf{e}^{(1)}$. As along the $\mathbf{e}^{(1)}$ direction $xy > 0$, the nonlinear term in the $\dot{z}$ equation (2.12) bends both branches of the EQ_0 unstable manifold $W^u(EQ_0)$ upwards. Then . . . - never mind. Best to postpone the completion of this narrative to example 9.2, where the discrete symmetry of Lorenz flow will help us streamline the analysis. As we shall show, what we already know about the 3 equilibria and their stable/unstable manifolds suffices to completely pin down the topology of Lorenz flow.

(Evangelos Siminos and Jonathan Halcrow)

4.3.2 Stability of trajectories

Next, consider the case of a general, non-stationary trajectory $x(t)$. The exponential of a constant matrix can be defined either by its Taylor series expansion, or in terms of the Euler limit (4.15):

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$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \tag{4.40}$$

$$= \lim_{m\to\infty} \left(\mathbf{1} + \frac{t}{m} A\right)^m . \tag{4.41}$$

Taylor expanding is fine if A is a constant matrix. However, only the second, tax-accountant's discrete step definition of an exponential is appropriate for the task at hand, as for a dynamical system the local rate of neighborhood distortion $A(x)$ depends on where we are along the trajectory. The linearized neighborhood is multiplicatively deformed along the flow, and the m discrete time step approximation to J^t is therefore given by a generalization of the Euler product (4.41):

$$J^t = \lim_{m\to\infty} \prod_{n=m}^{1} (\mathbf{1} + \delta t A(x_n)) = \lim_{m\to\infty} \prod_{n=m}^{1} e^{\delta t\, A(x_n)} \tag{4.42}$$

$$= \lim_{m\to\infty} e^{\delta t\, A(x_n)} e^{\delta t\, A(x_{m-1})} \cdots e^{\delta t\, A(x_2)} e^{\delta t\, A(x_1)} ,$$

where $\delta t = (t - t_0)/m$, and $x_n = x(t_0 + n\delta t)$. Slightly perverse indexing of the products indicates that in our convention the successive infinitesimal deformation are applied by multiplying from the left. The two formulas for J^t agree to leading order in δt, and the $m \to \infty$ limit of this procedure is the integral

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$$J^t_{ij}(x_0) = \left[\mathbf{T} e^{\int_0^t d\tau A(x(\tau))}\right]_{ij} , \tag{4.43}$$

where $\mathbf{T}$ stands for time-ordered integration, *defined* as the continuum limit of the successive left multiplications (4.42). This integral formula for J is the main conceptual result of this chapter.

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It makes evident important properties of fundamental matrices, such as that they are multiplicative along the flow,

$$J^{t+t'}(x) = J^{t'}(x')\, J^t(x), \qquad \text{where } x' = f^t(x) , \tag{4.44}$$

an immediate consequence of time-ordered product structure of (4.42). However, in practice J is evaluated by integrating (4.9) along with the ODEs that define a particular flow.

in depth:
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4.4 Neighborhood volume

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remark 15.3

Consider a small state space volume $\Delta V = d^d x$ centered around the point x_0 at time $t = 0$. The volume $\Delta V' = \Delta V(t)$ around the point $x' = x(t)$ time t later is

$$\Delta V' = \frac{\Delta V'}{\Delta V} \Delta V = \left|\det \frac{\partial x'}{\partial x}\right| \Delta V = \left|\det J(x_0)^t\right| \Delta V , \tag{4.45}$$

so the $|\det J|$ is the ratio of the initial and the final volumes. The determinant $\det J^t(x_0) = \prod_{i=1}^d \Lambda_i(x_0, t)$ is the product of the multipliers. We shall refer to this determinant as the *Jacobian* of the flow. This Jacobian is easily evaluated. Take the time derivative and use the matrix identity $\ln\det J = \mathrm{tr}\,\ln J$:

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$$\frac{d}{dt} \ln \Delta V(t) = \frac{d}{dt} \ln\det J = \mathrm{tr}\, \frac{d}{dt} \ln J = \mathrm{tr}\, \frac{1}{J} \dot{J} = \mathrm{tr}\, A = \partial_i v_i .$$

(Here, as elsewhere in this book, a repeated index implies summation.) As the divergence $\partial_i v_i$ is a scalar quantity, the integral in the exponent needs *no time ordering*. Integrate both sides to obtain the time evolution of an infinitesimal volume

$$\det J^t(x_0) = \exp\left[\int_0^t d\tau\, \mathrm{tr}\, \mathbf{A}(x(\tau))\right] = \exp\left[\int_0^t d\tau\, \partial_i v_i(x(\tau))\right] . \tag{4.46}$$

All we need to do is evaluate the time average

$$
\begin{aligned}
\overline{\partial_i v_i} &= \lim_{t\to\infty} \frac{1}{t}\int_0^t d\tau \sum_{i=1}^{d} A_{ii}(x(\tau)) \\
&= \frac{1}{t}\ln\left|\prod_{i=1}^{d}\Lambda_i(x_0,t)\right| = \sum_{i=1}^{d}\lambda^{(i)}(x_0,t)
\end{aligned}
\tag{4.47}
$$

along the trajectory. If the flow is not singular (for example, the trajectory does not run head-on into the Coulomb $1/r$ singularity), the stability matrix elements are bounded everywhere, $|A_{ij}| < M$, and so is the trace $\sum_i A_{ii}$. The time integral in (4.46) grows at most linearly with t, hence $\overline{\partial_i v_i}$ is bounded for all times, and numerical estimates of the $t \to \infty$ limit in (4.47) are not marred by any blowups.

Even if we were to insist on extracting $\overline{\partial_i v_i}$ from (4.42) by first multiplying fundamental matrices along the flow, and then taking the logarithm, we can avoid exponential blowups in J^t by using the multiplicative structure (4.44), $\det J^{t'+t}(x_0) = \det J^{t'}(x')\det J^t(x_0)$ to restart with $J^0(x') = \mathbf{1}$ whenever the eigenvalues of $J^t(x_0)$ start getting out of hand. In numerical evaluations of Lyapunov exponents, $\lambda_i = \lim_{t\to\infty}\mu^{(i)}(x_0,t)$, the sum rule (4.47) can serve as a helpful check on the accuracy of the computation.

sect. 15.3

The divergence $\partial_i v_i$ is an important characterization of the flow - it describes the behavior of a state space volume in the infinitesimal neighborhood of the trajectory. If $\partial_i v_i < 0$, the flow is *locally contracting*, and the trajectory might be falling into an attractor. If $\partial_i v_i(x) < 0$, for all $x \in \mathcal{M}$, the flow is *globally* contracting, and the dimension of the attractor is necessarily smaller than the dimension of state space $\mathcal{M}$. If $\partial_i v_i = 0$, the flow preserves state space volume and $\det J^t = \mathbf{1}$. A flow with this property is called *incompressible*. An important class of such flows are the Hamiltonian flows considered in sect. 6.2.

But before we can get to that, Henri Roux, the perfect student always on alert, pipes up. He does not like our definition of the fundamental matrix in terms of the time-ordered exponential (4.43). Depending on the signs of multipliers, the left hand side of (4.46) can be either positive or negative. But the right hand side is an exponential of a real number, and that can only be positive. What gives? As we shall see much later on in this text, in discussion of topological indices arising in semiclassical quantization, this is not at all a dumb question.

sect. 30.2

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4.5 Stability of maps

The transformation of an infinitesimal neighborhood of a trajectory under the iteration of a map follows from Taylor expanding the iterated mapping at *discrete* time n to linear order, as in (4.5). The linearized neighborhood is transported by the fundamental matrix evaluated at a discrete set of times $n = 1, 2, \ldots$,

$$
M^n_{ij}(x_0) = \left.\frac{\partial f^n_i(x)}{\partial x_j}\right|_{x=x_0} . \tag{4.48}
$$

We shall refer to this Jacobian matrix also as the *monodromy* matrix, in case of periodic orbits $f^n(x) = x$. Derivative notation $M^t(x_0) \to Df^t(x_0)$ is frequently employed in the literature. As in the continuous case, we denote by Λ_k the kth *eigenvalue* or multiplier of the finite time fundamental matrix $M^n(x_0)$, and $\mu^{(k)}$ the real part of kth *eigen-exponent*

$$
\Lambda_\pm = e^{n(\mu\pm i\omega)} , \qquad |\Lambda| = e^{n\mu} .
$$

For complex eigenvalue pairs the phase ω describes the rotation velocity in the plane defined by the corresponding pair of eigenvectors, with one period of rotation given by

$$
T = 2\pi/\omega . \tag{4.49}
$$

Example 4.8 Stability of a 1-dimensional map: *Consider a 1-d map $f(x)$. The chain rule yields the stability of the nth iterate*

$$
\Lambda(x_0,n) = \frac{d}{dx}f^n(x_0) = \prod_{m=0}^{n-1} f'(x_m) , \quad x_m = f^m(x_0) . \tag{4.50}
$$

The 1-step product formula for the stability of the nth iterate of a d-dimensional map

$$
\begin{aligned}
M^n(x_0) &= M(x_{n-1})\cdots M(x_1)M(x_0) , \\
M(x)_{kl} &= \frac{\partial}{\partial x_l} f_k(x) , \quad x_m = f^m(x_0)
\end{aligned}
\tag{4.51}
$$

follows from the chain rule for matrix derivatives

$$
\frac{\partial}{\partial x_i} f_j(f(x)) = \sum_{k=1}^{d} \left.\frac{\partial}{\partial y_k} f_j(y)\right|_{y=f(x)} \frac{\partial}{\partial x_i} f_k(x) .
$$

If you prefer to think of a discrete time dynamics as a sequence of Poincaré section returns, then (4.51) follows from (4.44): fundamental matrices are multiplicative along the flow.

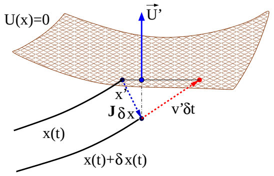

Figure 4.8: If $x(t)$ intersects the Poincaré section $\mathcal{P}$ at time τ, the nearby $x(t) + \delta x(t)$ trajectory intersects it time $\tau+\delta t$ later. As $(U' \cdot v'\delta t) = -(U' \cdot J\,\delta x)$, the difference in arrival times is given by $\delta t = -(U' \cdot J\,\delta x)/(U' \cdot v')$.

Example 4.9 Hénon map fundamental matrix: *For the Hénon map (3.15) the fundamental matrix for the nth iterate of the map is*

$$M^n(x_0) = \prod_{m=n}^{1} \begin{pmatrix} -2ax_m & b \\ 1 & 0 \end{pmatrix}, \qquad x_m = f_1^m(x_0, y_0)\,. \tag{4.52}$$

The determinant of the Hénon one time step fundamental matrix (4.52) is constant,

$$\det M = \Lambda_1 \Lambda_2 = -b \tag{4.53}$$

so in this case only one eigenvalue $\Lambda_1 = -b/\Lambda_2$ needs to be determined. This is not an accident; a constant Jacobian was one of desiderata that led Hénon to construct a map of this particular form.

fast track:
chapter 6, p. 99

4.5.1 Stability of Poincaré return maps

(R. Paškauskas and P. Cvitanović)

We now relate the linear stability of the Poincaré return map $P : \mathcal{P} \to \mathcal{P}$ defined in sect. 3.1 to the stability of the continuous time flow in the full state space.

The hypersurface $\mathcal{P}$ can be specified implicitly through a function $U(x)$ that is zero whenever a point x is on the Poincaré section. A nearby point $x + \delta x$ is in the hypersurface $\mathcal{P}$ if $U(x + \delta x) = 0$, and the same is true for variations around the first return point $x' = x(\tau)$, so expanding $U(x')$ to linear order in δx leads to the condition

$$\sum_{i=1}^{d+1} \frac{\partial U(x')}{\partial x_i} \left.\frac{dx'_i}{dx_j}\right|_{\mathcal{P}} = 0\,. \tag{4.54}$$

In what follows U_i is the gradient of U defined in (3.3), unprimed quantities refer to the starting point $x = x_0 \in \mathcal{P}$, $v = v(x_0)$, and the primed quantities to the first return: $x' = x(\tau)$, $v' = v(x')$, $U' = U(x')$. For brevity we shall also denote the full state space fundamental matrix at the first return by $J = J^\tau(x_0)$. Both the first return x' and the time of flight to the next

Poincaré section $\tau(x)$ depend on the starting point x, so the fundamental matrix

$$\hat{J}(x)_{ij} = \left.\frac{dx'_i}{dx_j}\right|_{\mathcal{P}} \tag{4.55}$$

with both initial and the final variation constrained to the Poincaré section hypersurface $\mathcal{P}$ is related to the continuous flow fundamental matrix by

$$\left.\frac{dx'_i}{dx_j}\right|_{\mathcal{P}} = \frac{\partial x'_i}{\partial x_j} + \frac{dx'_i}{d\tau}\frac{d\tau}{dx_j} = J_{ij} + v'_i \frac{d\tau}{dx_j}\,.$$

The return time variation $d\tau/dx$, figure 4.8, is eliminated by substituting this expression into the constraint (4.54),

$$0 = \partial_i U'\, J_{ij} + (v' \cdot \partial U')\frac{d\tau}{dx_j}\,,$$

yielding the projection of the full space $(d + 1)$-dimensional fundamental matrix to the Poincaré map d-dimensional fundamental matrix:

$$\hat{J}_{ij} = \left(\delta_{ik} - \frac{v'_i\, \partial_k U'}{(v' \cdot \partial U')}\right) J_{kj}\,. \tag{4.56}$$

Substituting (4.7) we verify that the initial velocity $v(x)$ is a zero-eigenvector of $\hat{J}$

$$\hat{J}v = 0\,, \tag{4.57}$$

so the Poincaré section eliminates variations parallel to v, and $\hat{J}$ is a rank d matrix, i.e., one less than the dimension of the continuous time flow.

Résumé

A neighborhood of a trajectory deforms as it is transported by a flow. In the linear approximation, the stability matrix A describes the shearing/-compression/expansion of an infinitesimal neighborhood in an infinitesimal time step. The deformation after a finite time t is described by the fundamental matrix

$$J^t(x_0) = \mathbf{T} e^{\int_0^t d\tau A(x(\tau))}\,,$$

where $\mathbf{T}$ stands for the time-ordered integration, defined multiplicatively along the trajectory. For discrete time maps this is multiplication by time step fundamental matrix M along the n points $x_0, x_1, x_2, \ldots, x_{n-1}$ on the trajectory of x_0,

$$M^n(x_0) = M(x_{n-1})M(x_{n-2})\cdots M(x_1)M(x_0)\,,$$

with $M(x)$ the single discrete time step fundamental matrix. In this book Λ_k denotes the kth *eigenvalue* of the finite time fundamental matrix $J^t(x_0)$, and $\mu^{(k)}$ the real part of kth *eigen-exponent*

$$|\Lambda| = e^{n\mu}\,, \qquad \Lambda_\pm = e^{n(\mu \pm i\omega)}\,.$$

For complex eigenvalue pairs the phase ω describes rotational motion in the plane defined by the corresponding pair of eigenvectors.

The eigenvalues and eigen-directions of the fundamental matrix describe the deformation of an initial infinitesimal sphere of neighboring trajectories into an ellipsoid a finite time t later. Nearby trajectories separate exponentially along unstable directions, approach each other along stable directions, and change slowly (algebraically) their distance along marginal directions. The fundamental matrix J^t is in general neither symmetric, nor diagonalizable by a rotation, nor do its (left or right) eigenvectors define an orthonormal coordinate frame. Furthermore, although the fundamental matrices are multiplicative along the flow, in dimensions higher than one their eigenvalues in general are not. This lack of multiplicativity has important repercussions for both classical and quantum dynamics.

☞ appendix G.1

Commentary

Remark 4.1 Linear flows. The subject of linear algebra generates innumerable tomes of its own; in sect. 4.2 we only sketch, and in appendix B recapitulate a few facts that our narrative relies on. They are presented at length in many textbooks. The standard references that exhaustively enumerate and explain all possible cases are Hirsch and Smale [4.1], and Arnol'd [7.1]. For ChaosBook purposes, we enjoyed the discussion in chapter 2 Meiss [4.2], chapter 1 of Perko [4.3] and chapters 3 and 5 of Glendinning [4.4] most.

The construction of projection operators given here is taken from refs. [**?**, **?**]. Who wrote this down first we do not know, lineage certainly goes all the way back to Lagrange polynomials [4.6], but projection operators tend to get drowned in sea of algebraic details. Halmos [4.5] is a good early reference - but we like Harter's exposition [**?**, **?**, 4.8] best, for its multitude of specific examples and physical illustrations.

The nomenclature tends to be a bit confusing. Sometimes A, the stability matrix (4.3) which describes the instantaneous shear of the trajectory point $x(x_0, t)$ is referred to as the 'Jacobian matrix,' a particularly unfortunate usage when one considers linearized stability of an equilibrium point (4.30). What Jacobi had in mind in his 1841 fundamental paper [4.7] on the determinants today known

as 'jacobians' were transformations between different coordinate frames. More unfortunate still is referring to $J^t = e^{tA}$ as an 'evolution operator,' which here (see sect. 15.2) refers to something altogether different. In this book fundamental matrix J^t always refers to (4.6), the linearized deformation after a finite time t, either for a continuous time flow, or a discrete time mapping.

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