ChaosBook.org chapter

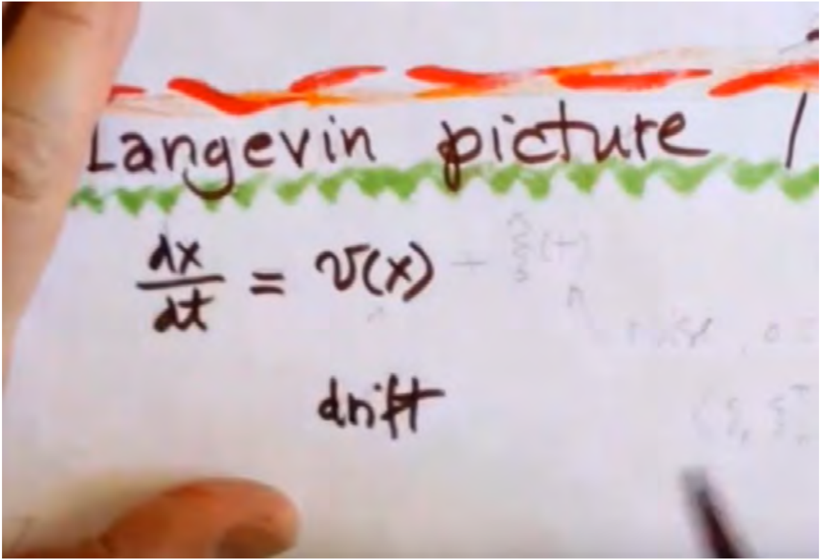
noise

July 18, 2025 version 17.6.8,

Outline

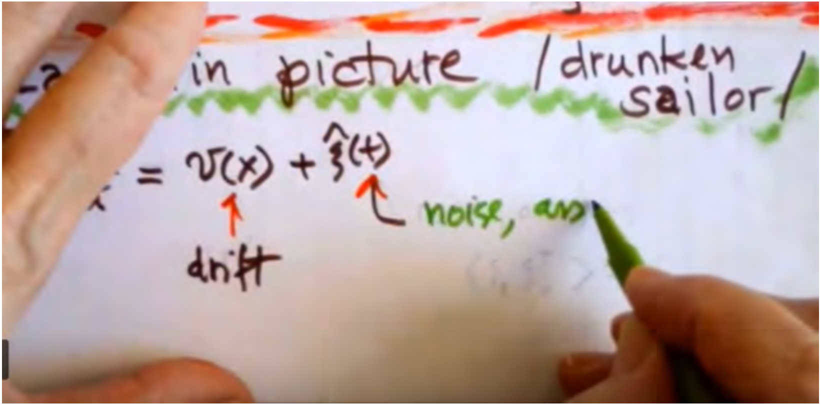
1. **what is “noise”**
   - Langevin
   - Fokker-Planck
   - Diffusion
2. **what this chapter is about**
   - knowing when to stop
3. **dynamicist’s view of noise**
   - dynamicist’s view of noise
   - idea #1: evolve densities, not noisy trajectories
   - one more time, now with math
4. **nobody understands chaos :(**
   - idea #2: for unstable directions, look back
   - idea #3: partition by periodic points
   - one more time, now with math
5. **optimal partition hypothesis**

who’s your friend ?



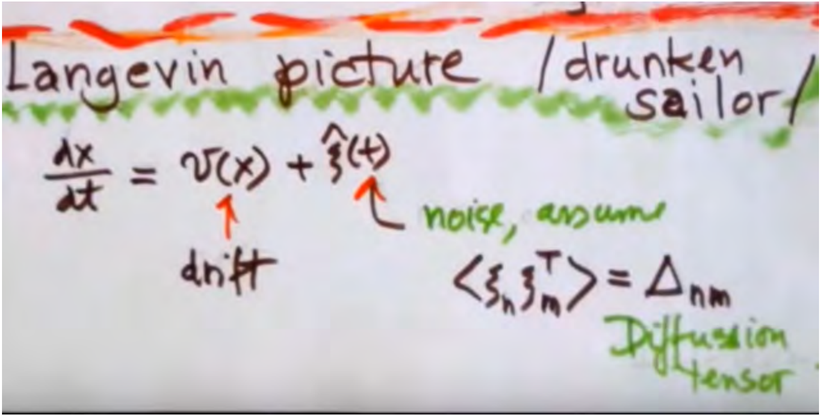
Chiara’s your friend !

how old is your friend ?



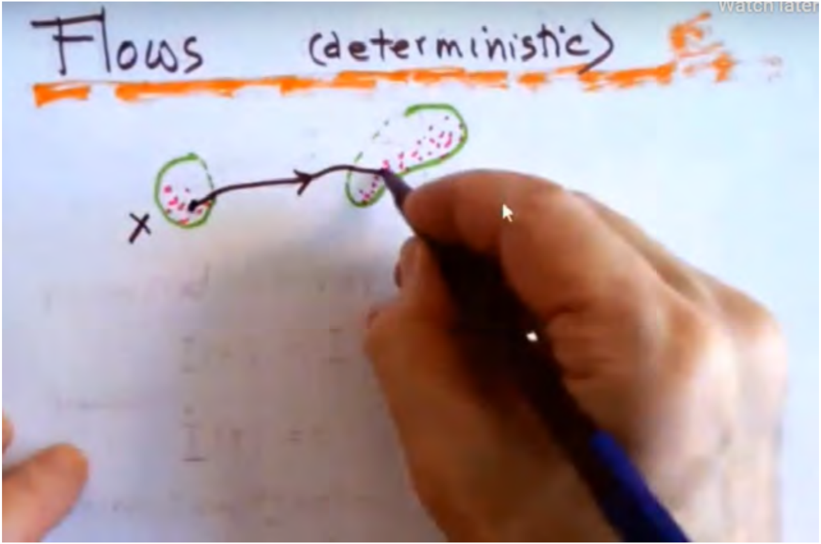
Chiara’s is 25+ !

what loves kinky trajectories?



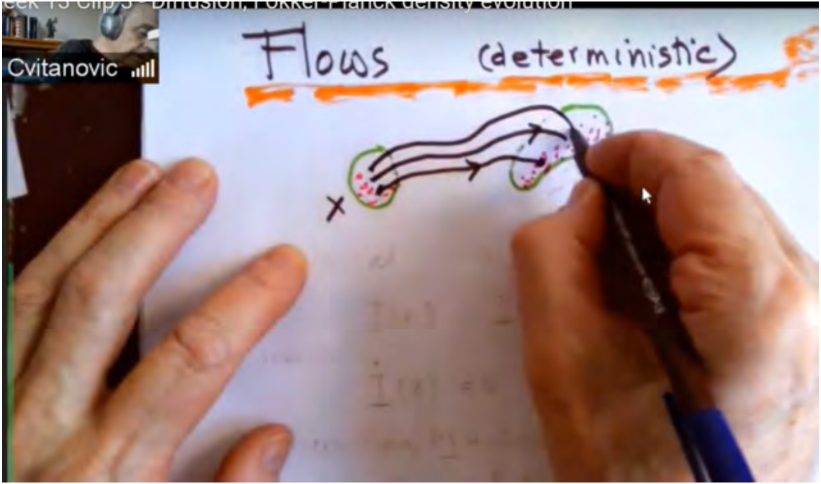
Chemists love kinky trajectories !

who’s your friend ?



Inés is your friend !

how old is your friend ?



Inés is 55 !

dynamical system

state space

a manifold $\mathcal{M} \in \mathbb{R}^d$: d numbers determine the state of the system

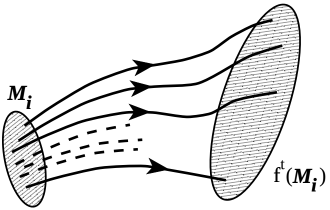
representative point

$x(t) \in \mathcal{M}$
a state of physical system at instant in time

dynamics

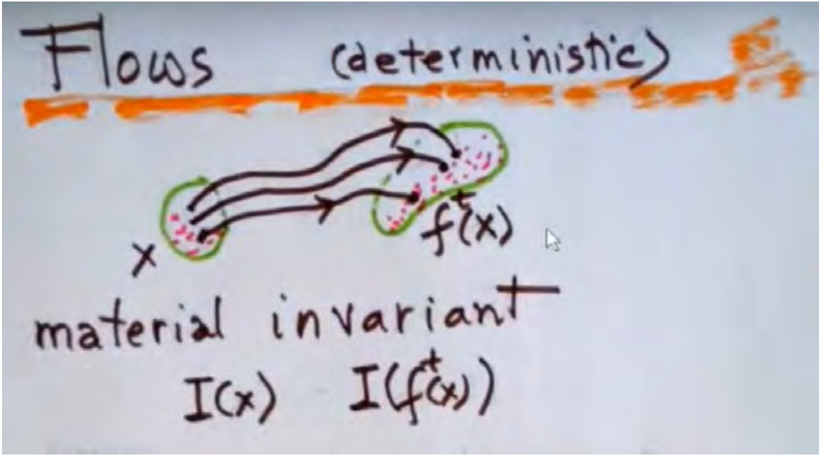
map $f^t(x_0)$ = representative point time t later

evolution in time



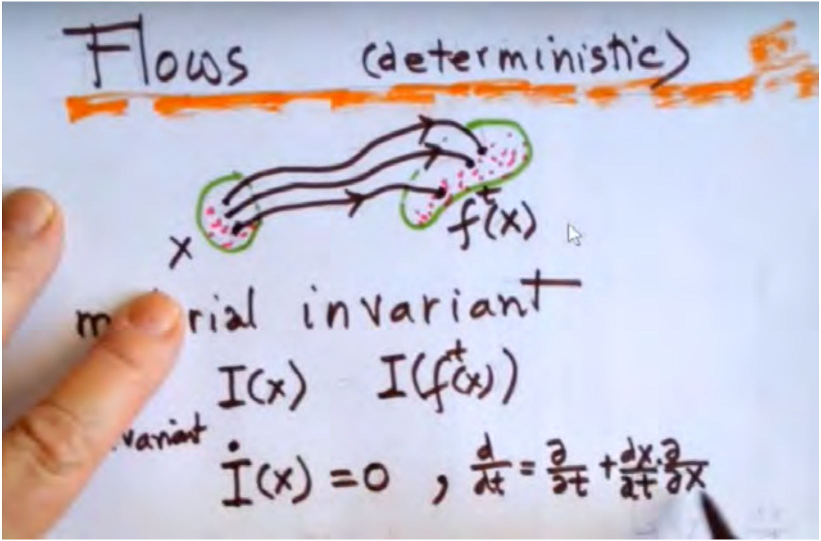
f^t maps a region $\mathcal{M}_i$ of the state space into the region $f^t(\mathcal{M}_i)$

what does your friend have ?



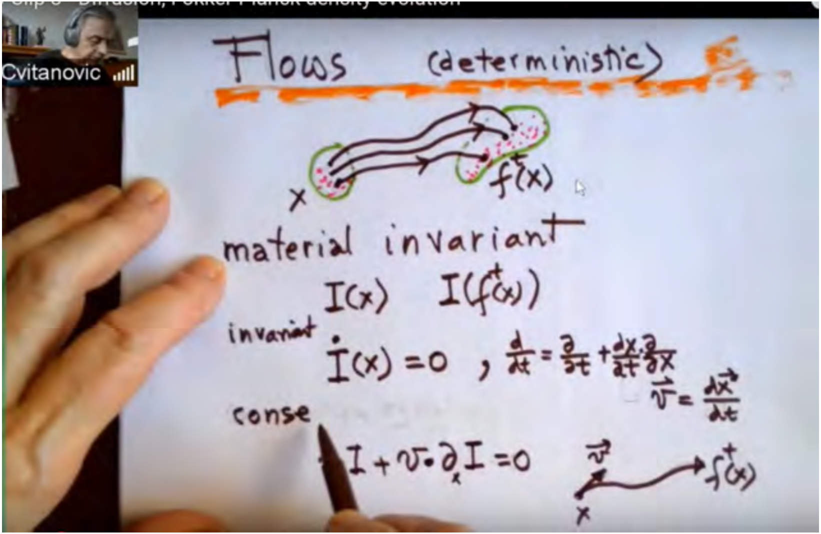
Inés's got information !

who are you afraid of ?



noise !

who are you afraid of ?



noise !

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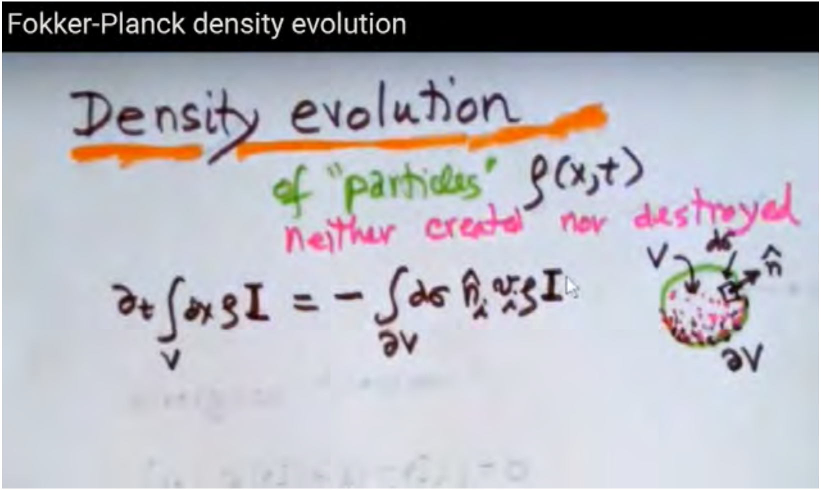
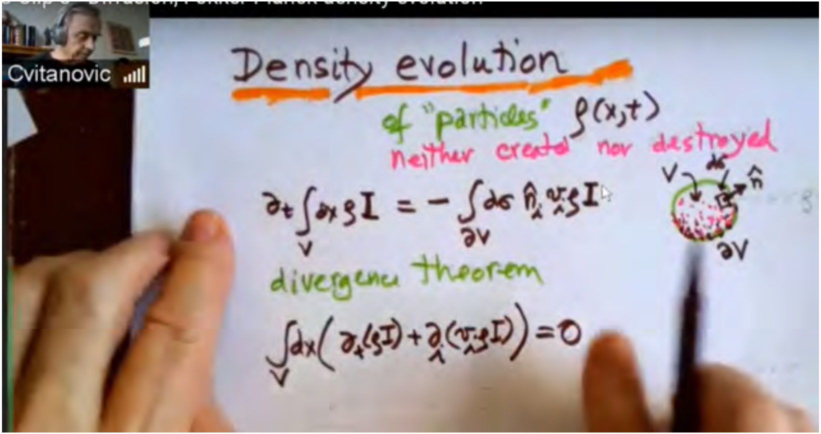


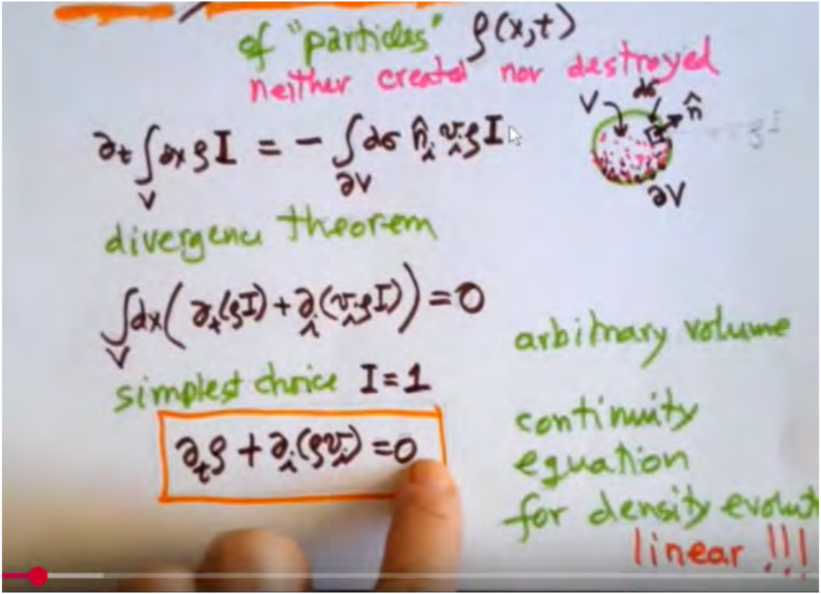
noise !

who are you afraid of ?



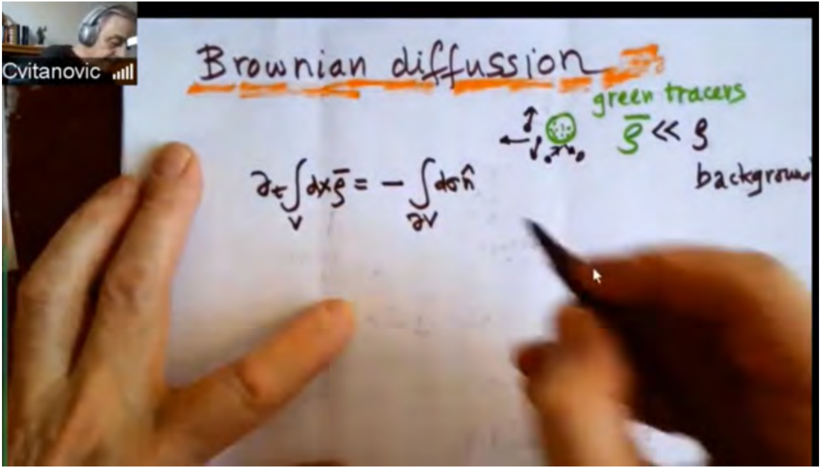
Fokker-Planck !

Newton equations are nonlinear



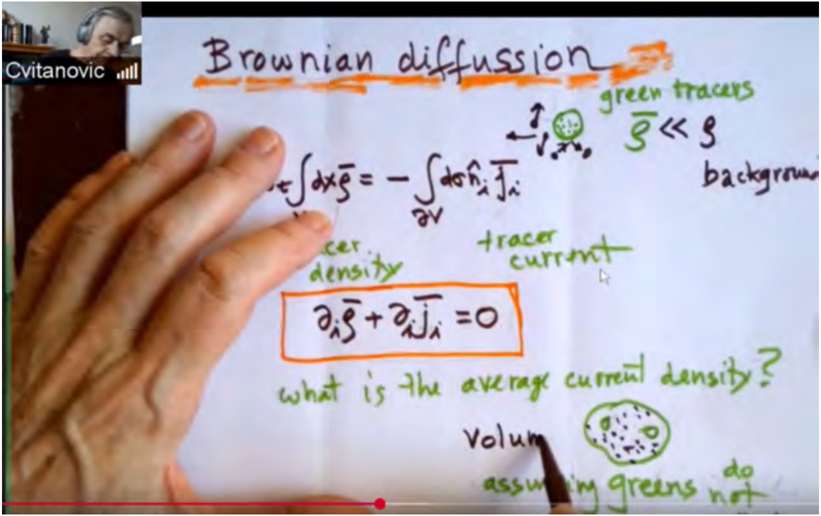
equations for density transport are linear !

if space is homogenous and isotropic



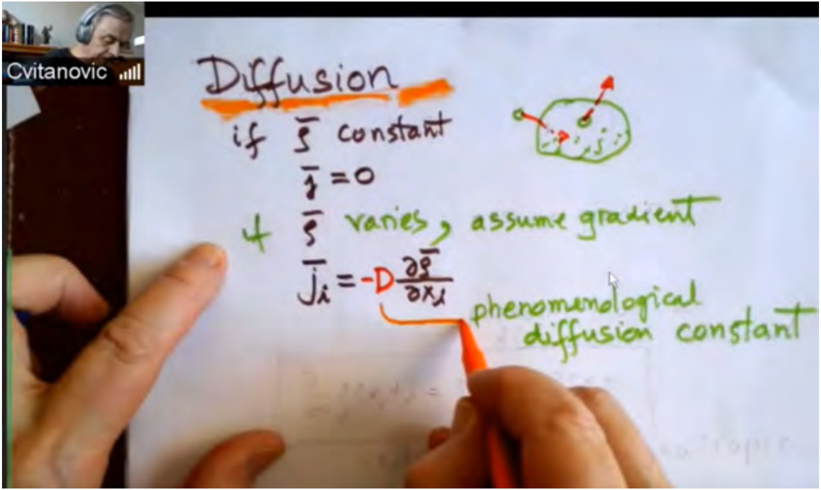
cannot tell left from right

who are you afraid of ?



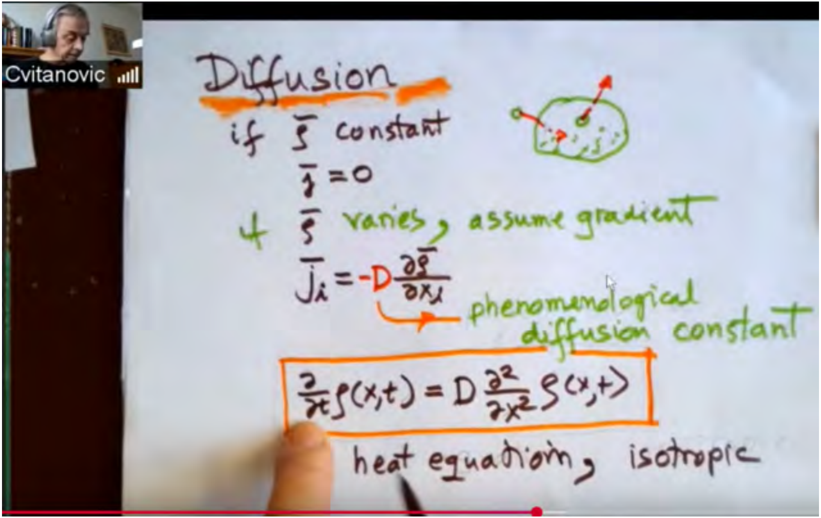
noise !

which way are you going to go ?



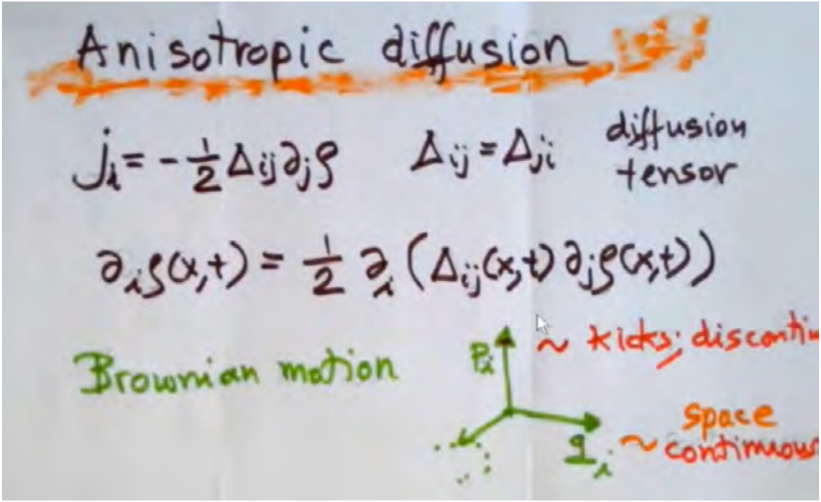
along gradient, away from the madding crowd

can't tell left from right ?



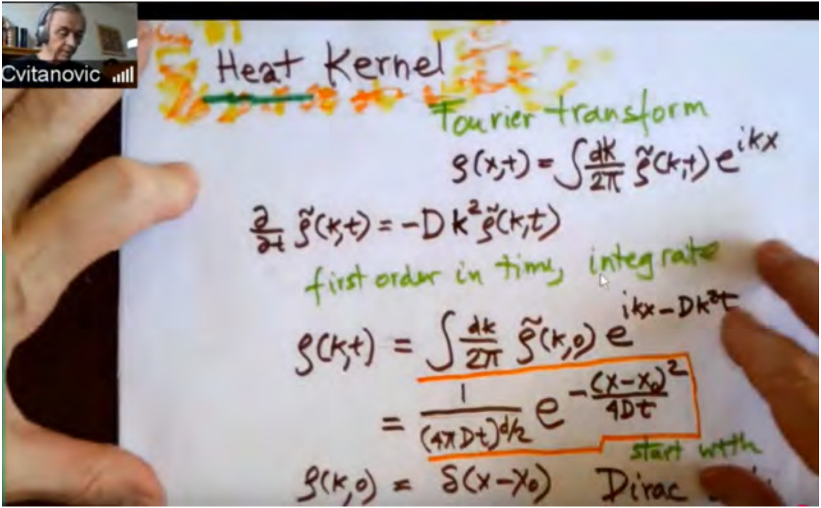
Laplacian !

not all directions are equal?



continuous trajectory, discontinuous kicks !

who are you in awe of ?



Einstein !

standard normal (Gaussian) probability distribution

d-dimensional *discrete time* stochastic flow

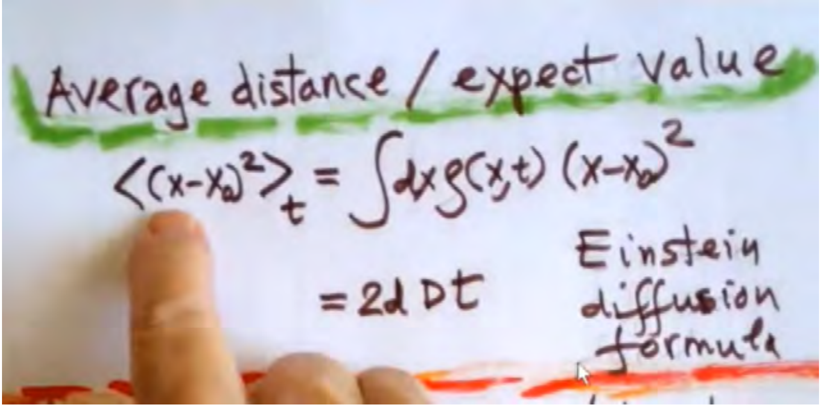
$$x' = f(x) + \xi_a$$

1-time step evolution = probability of reaching x' given random kick, Gaussian distributed $\xi_a = x' - f(x)$

$$\frac{1}{\sqrt{4\pi D}} \exp\left(-\frac{\xi_a^2}{4D}\right)$$

variance $2D$, standard deviation $\sqrt{2D}$

can’t tell left from right ?



Laplacian !

sources

so far :

noise, no dynamics

- Langevin : kinky trajectories. nonlinear.
- Fokker-Planck : smooth densities. linear.

sources

so far :

noise, no dynamics

- Langevin : kinky trajectories. nonlinear.
- Fokker-Planck : smooth densities. linear.
- "Mathematical Methods of Physics" week 13

sources

so far :

noise, no dynamics

- Langevin : kinky trajectories. nonlinear.
- Fokker-Planck : smooth densities. linear.
- "Mathematical Methods of Physics" week 13
- ChaosBook.org "Noise" overheads, videos
- ChaosBook.org chapter 33 "Noise"

3rd millennium problem(s)

you have a brain

neuron i activity = axis x_i : 86 billion neurons

experiment ⇒ neural manifolds data

embedded in state space : many low-dimensional manifolds

data ⇒ neural manifold models

each can be modeled as a dynamical system

[1,2]

but neurons are very noisy ?

that's a good thing :)

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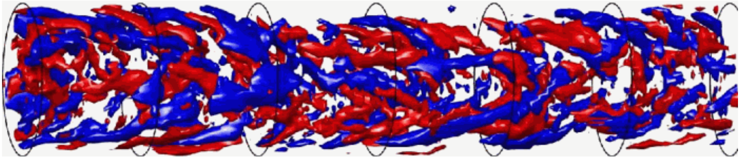
[1] J. A. Gallego et al., Nat. Commun. **9**, 4233 (2018).

[2] J. Ladenbauer et al., Nat. Commun. **10**, 4933 (2019).

dynamical theory of turbulence?

computation of unstable periodic orbits in high-dimensional state spaces, such as Navier-Stokes,



is at the border of what is feasible numerically, and criteria to identify finite sets of the most important solutions are very much needed.

when are we to stop calculating these solutions?

fugget about it

this is way too much work !

noise is your friend

1. dynamicist's view of noise
2. be bore : fall into a hole
3. live life : chaos rules
4. optimal partition hypothesis

knowing when to stop

need the 3D velocity field at every (x, y, z)!

motions of fluids : require ∞ bits?

numerical simulations track 10^2 - 10^6 of computational degrees of freedom; terabytes of data, but how much information is there in all of this?

knowing when to stop

motions of fluids : require ∞ bits??

that cannot be right...

knowing when to stop

Science originates from curiosity and bad eyesight.

— Bernard de Fontenelle,
Entretiens sur la Pluralité des Mondes Habités

in practice

every physical problem is coarse partitioned and finite

noise rules the state space

- any physical system experiences (some kind of) noise
- any numerical computation is ‘noisy’
- any prediction only needs a desired finite accuracy

noisy dynamics

stochastic dynamical system

the triple $(\mathcal{M}, f, \Delta)$

where $\Delta(x)$ is the noise covariance matrix

the problem

enumerate, classify all solutions of $(\mathcal{M}, f, \Delta)$

including $Q(x_j)$, the density covariance matrix

all worked out in 1810

Laplace, P. S.

Mémoire sur les intégrales définies et leur application aux probabilités, et spécialement à la recherche du milieu qu'il faut choisir entre les résultats des observations,

Mem. Acad. Sci. (I), XI, Sec. V., 375–387 (1810)

OK, I have not read it, so ...

how big is the neighborhood blurred by the accumulated noise?

the (well known) **key formula** that we now derive:

$$Q_{n+1} = M_n Q_n M_n^T + \Delta_n$$

density covariance matrix at time n: Q_n
noise covariance matrix: Δ_n
Jacobian matrix of linearized flow: M_n

Lyapunov equation, doctoral dissertation 1892
Ornstein-Uhlenbeck 1930
Kalman filter 'prediction' 1960

derivation

keep things simple: illustrate by

d-dimensional *discrete time* stochastic flow

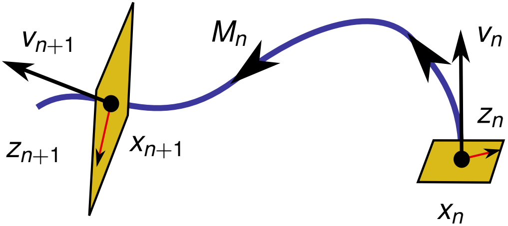
$$x_{a+1} = f(x_a) + \xi_a$$

uncorrelated in time

$$\langle \xi_a \rangle = 0 \,, \qquad \langle \xi_a \cdot \xi_b \rangle = 2\, d\, D\, \delta_{ab}$$

[all results apply both to the continuous and discrete time flows]

linearized deterministic flow



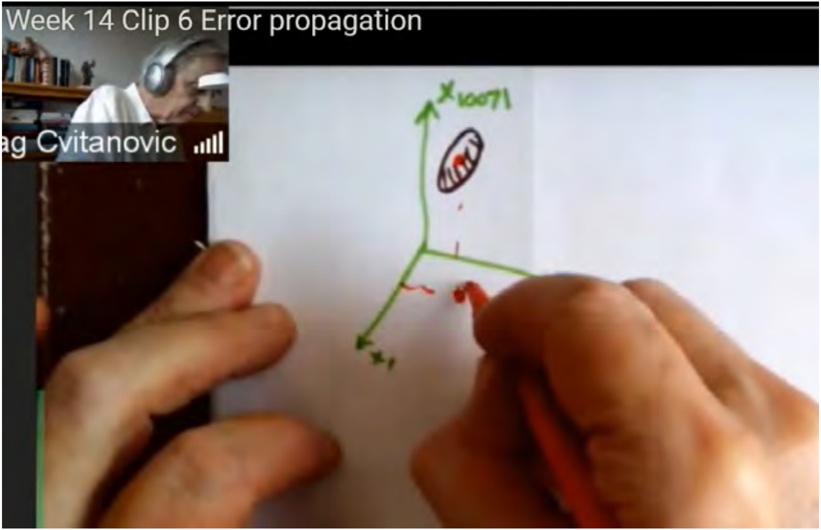
$$x_{n+1} + z_{n+1} = f(x_n) + M_n\, z_n\,, \quad M_{ij} = \partial f_i / \partial x_j$$

in one time step a linearized neighborhood of x_n is

(1) advected by the flow

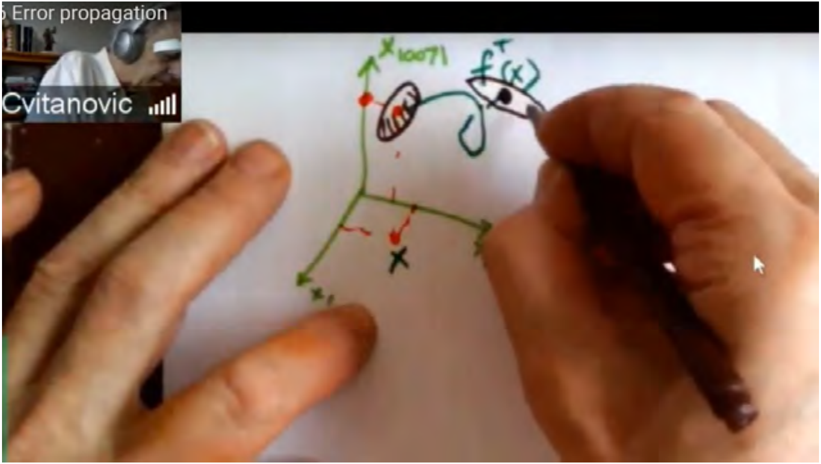
(2) transported by the Jacobian matrix M_n into a neighborhood given by the M eigenvalues and eigenvectors

who’s your friend ?



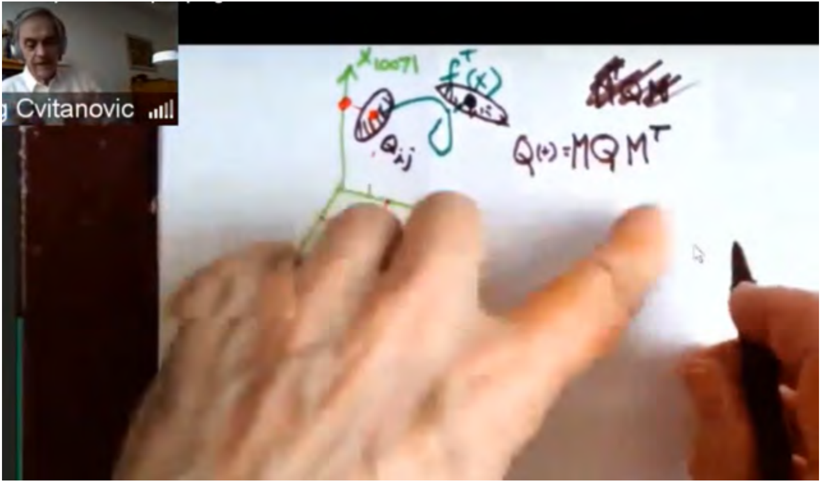
Sara’s your friend !

how old is your friend ?



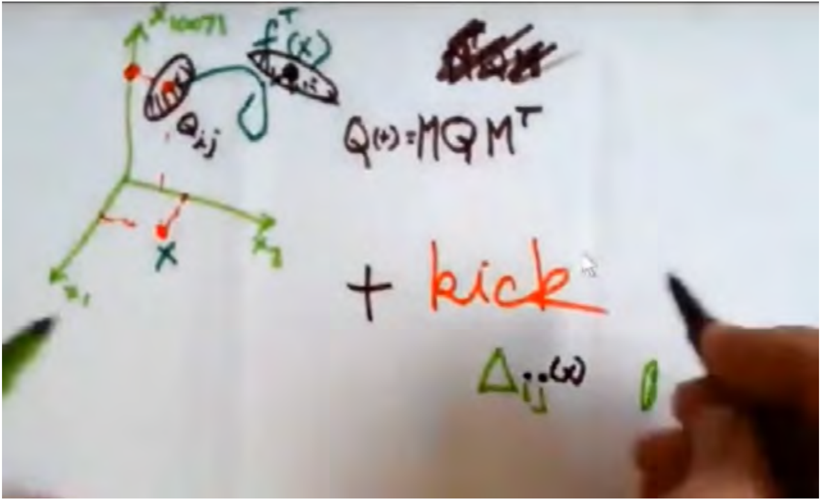
Sara’s is 75 !

who’s wiser ?



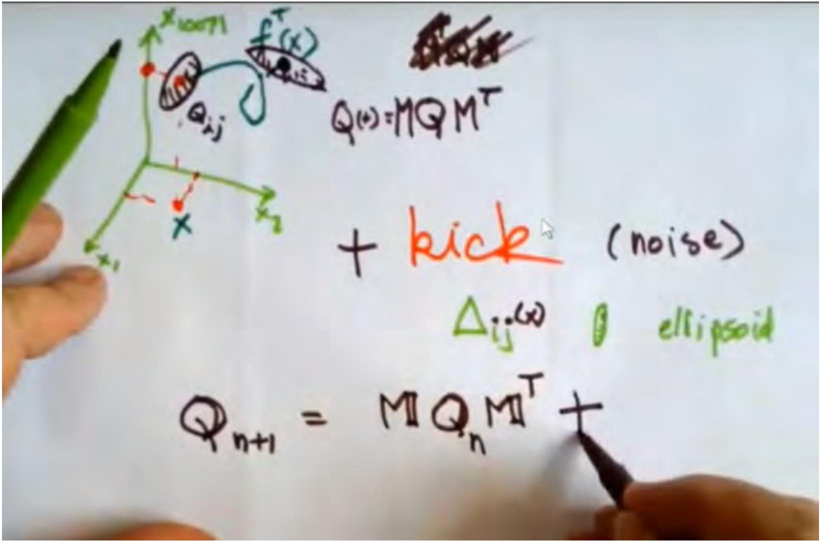
|Chiara| + |Inés| $\geq$ |Sara| !

who’s your friend ?



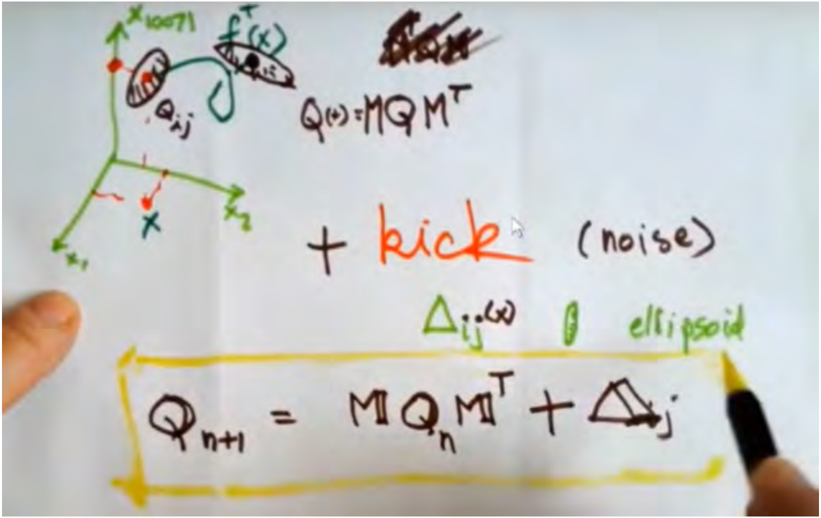
noise is your friend !

who’s your friend ?



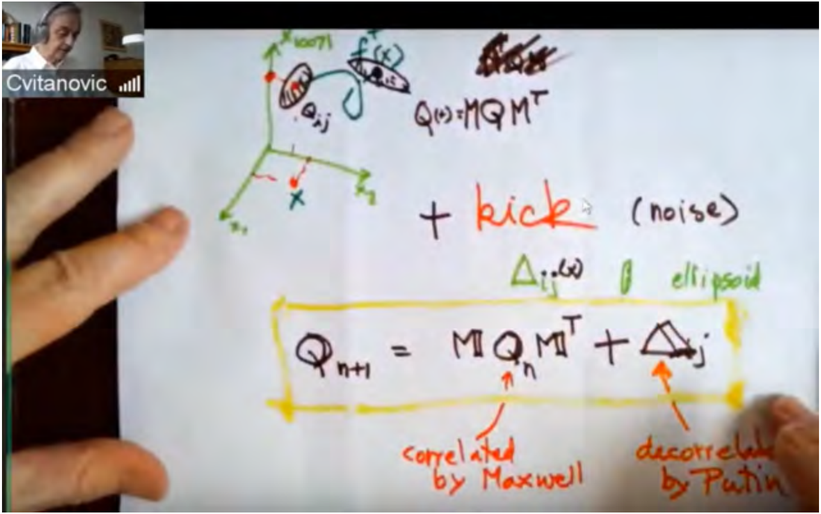
noise is your friend !

who’s your friend ?



noise is your friend !

who's your friend ?



noise is your friend !

covariance advection

let the initial density of deviations z from the deterministic center be a Gaussian whose covariance matrix is

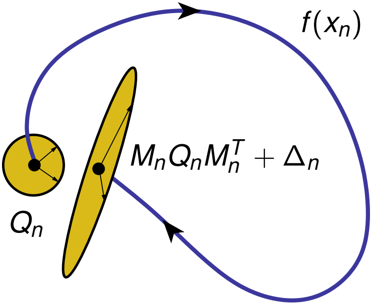
$$Q_{jk} = \langle z_j z_k^T \rangle$$

a step later the Gaussian is advected to

$$\begin{aligned} \langle z_j z_k^T \rangle &\rightarrow \langle (M\,z)_j\,(M\,z)_k^T \rangle \\ Q &\rightarrow M\,Q\,M^T \end{aligned}$$

next: add noise

roll your own cigar



in one time step
a Gaussian density distribution with covariance matrix Q_n is

(1) advected by the flow

(2) smeared with additive noise

into a Gaussian 'cigar' whose widths and orientation are given by the singular values and vectors of Q_{n+1}

covariance evolution

$$Q_{n+1} = M_n Q_n M_n^T + \Delta_n$$

(1) advect deterministically
local density covariance matrix $Q \to MQM^T$

(2) add noise covariance matrix Δ

covariances add up as sums of squares

noise along a trajectory

iterate $Q_{a+1} = M_a Q_a M_a^T + \Delta_a$ along the trajectory

if M is contracting, over time the memory of the covariance Q_{a-n} of the starting density is lost, with iteration leading to the limit distribution

$$Q_a = \Delta_a + M_{a-1}\Delta_{a-1}M_{a-1}^T + M_{a-2}^2\Delta_{a-2}(M_{a-2}^2)^T + \cdots .$$

diffusive dynamics of a nonlinear system is fundamentally different from Brownian motion, as the flow induces a history dependent effective noise. Always

example : noise and a single attractive fixed point

if all eigenvalues of M are strictly contracting, all $|\Lambda_j| < 1$

any initial compact measure converges to the unique invariant Gaussian measure $\rho_0(z)$ whose covariance matrix satisfies

Lyapunov equation: time-invariant measure condition

$$Q = MQM^T + \Delta$$

[A. M. Lyapunov doctoral dissertation 1892]

solving for stationary covariance Q

assume that $[d \times d]$ matrix M has only nonzero eigenvalues $\{\Lambda_j\}$ and d linearly independent right and left eigenvectors (M is not defective)

$$M\,\mathbf{e}^{(j)} = \Lambda_j\,\mathbf{e}^{(j)}\,, \qquad \mathbf{e}_{(j)}\,M = \Lambda_j\,\mathbf{e}_{(j)}$$

eigenvectors can always be rescaled so that they are mutually orthogonal

$$\mathbf{e}_{(j)} \cdot \mathbf{e}^{(k)} = \delta_{jk}$$

form from the d column eigenvectors a $[d \times d]$ matrix

$$S = \left[\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \cdots, \mathbf{e}^{(d)}\right] , \qquad MS = \Lambda S$$

by $\mathbf{e}_{(j)} \cdot \mathbf{e}^{(k)} = \delta_{jk}$, the matrix whose rows are left eigenvectors is then the inverse

$$S^{-1} = \left[\mathbf{e}_{(1)}, \mathbf{e}_{(2)}, \cdots, \mathbf{e}_{(d)}\right]^T$$

S diagonalizes M and its transpose M^T by

similarity transformation

$$S^{-1} M S = \Lambda , \qquad S^T M^T (S^{-1})^T = \Lambda$$

define $\hat{Q} = S^{-1} Q \, (S^{-1})^T$ and $\hat{\Delta} = S^{-1} \Delta \, (S^{-1})^T$

time-invariant measure condition $Q = MQM^T + \Delta$ now takes form

$$\hat{Q} - \Lambda \hat{Q} \Lambda = \hat{\Delta}$$

matrix elements are $\hat{Q}_{ij}(1 - \Lambda_i \Lambda_j) = \hat{\Delta}_{ij}$, so

$$\hat{Q}_{ij} = \frac{\hat{\Delta}_{ij}}{1 - \Lambda_i \Lambda_j}$$

and the attracting fixed point covariance matrix is given by

$$Q = S \hat{Q} S^T$$

note!

covariance matrix

$$\hat{Q}_{ij} = \frac{\hat{\Delta}_{ij}}{1 - \Lambda_i \Lambda_j}$$

elements must be strictly positive

true only if all Floquet multipliers (Jacobian matrix M eigenvalues) are contracting, $|\Lambda_j| < 1$

summary: covariance matrix Q for an attractive fixed point

- determine the Jacobian matrix M eigenvalues and eigenvectors

$$M\,\mathbf{e}^{(j)} = \Lambda_j\,\mathbf{e}^{(j)}$$

- go to coordinate frame where M is diagonal,

$$S^{-1}MS = \Lambda\,, \qquad \hat{Q} = S^{-1}Q\,(S^{-1})^T\,, \qquad \hat{\Delta} = S^{-1}\Delta\,(S^{-1})^T$$

- evaluate

$$\hat{Q}_{ij} = \frac{\hat{\Delta}_{ij}}{1-\Lambda_i\Lambda_j}$$

- go back to the original coordinates

$$Q = S\hat{Q}S^T$$

a numerical diagonalization of the covariance matrix $Q = S\hat{Q}S^T$ yields the principal axis of the equilibrium Gaussian 'cigar'

eigenvectors of Q (it is a symmetric matrix) are orthogonal and have orientations distinct from the left/right eigenvectors of the non-normal Jacobian matrix M

1D example : Ornstein-Uhlenbeck process

contracting noisy 1-dimensional map

$$z_{n+1} = \Lambda z_n + \xi_n \,, \qquad |\Lambda| < 1$$

width of the natural measure[3] concentrated at the deterministic fixed point $z = 0$

$$Q = \frac{2D}{1 - |\Lambda|^2} \,, \qquad \rho_0(z) = \frac{1}{\sqrt{2\pi\, Q}} \exp\left(-\frac{z^2}{2\, Q}\right) \,,$$

[3] G. E. Uhlenbeck and L. S. Ornstein, Phys. Rev. **36**, 823–841 (1930).

1D example : Ornstein-Uhlenbeck process

width of the natural measure[3] concentrated at the deterministic fixed point $z = 0$

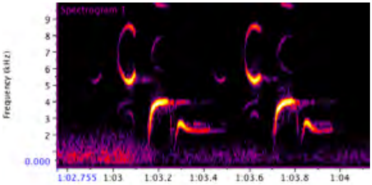
$$Q = \frac{2D}{1 - |\Lambda|^2}, \qquad \rho_0(z) = \frac{1}{\sqrt{2\pi\, Q}} \exp\left(-\frac{z^2}{2\,Q}\right),$$

- is balance between contraction by Λ and diffusive smearing by $2D$ at each time step
- for strongly contracting Λ, the width is due to the noise only
- As $|\Lambda| \to 1$ the width diverges: the trajectories are no longer confined, but diffuse by Brownian motion

[3] G. E. Uhlenbeck and L. S. Ornstein, Phys. Rev. **36**, 823–841 (1930).

high-dimensional example : birdsong in 'space'-time

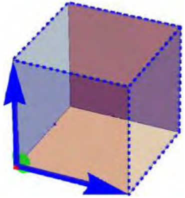
a song spectrogram evolving in time



Oak Titmouse

each pixel = a data point

a song is a point X in



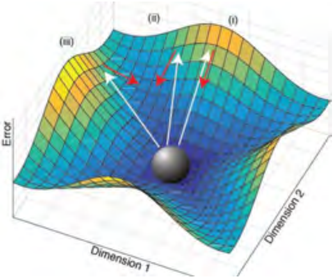
in the d^{LT}-dimensional hypercube state space

d bits at each of LT lattice sites

noise and a single attractive fixed point

an application to birdsong learning[4,5]

measure : covariance Q



learn : by poking

white arrows : kicking the song
red arrows : learning response

$$Q = MQM^T + \Delta$$

$\Rightarrow$ linearized dynamics matrix M

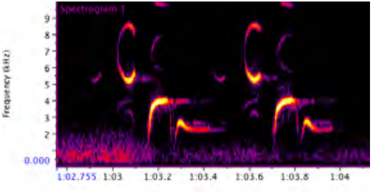
[4] S. A. Solla et al., in APS Meeting Abstracts (2016).

[5] Y. Cohen et al., in Comp. Systems Neuroscience (Cosyne), Lisbon, Portugal, edited by E. C. et. al. (2021).

‘space’-time field is ∞ dimensional ?"

a birdsong

in spacetime



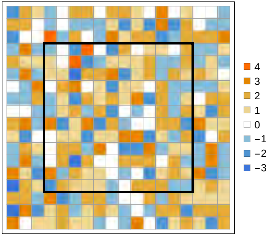
Oak Titmouse

each pixel = lattice site

neural manifold : d bits,
a few discretization sites

not a birdsong

This



is salt & pepper in the
d^{LT}-dimensional state space

periodic orbit example : 2D Brusselator limit cycle

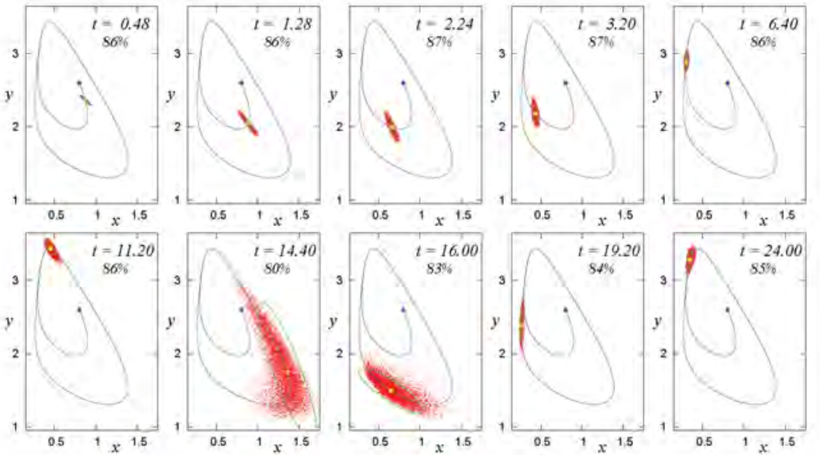


FIG. 2. Time development of distribution for Brusselator. 10 000 samples of Monte Carlo simulations are plotted by the red dots along with the covariance matrix $\hat{M}$ estimated by Eq. (E7); $\hat{M}$'s are represented by the green ellipses given by $\delta x^T \hat{M}^{-1} \delta x = 4/\Omega$, where $\delta x^T \equiv (x - x^*(t), y - y^*(t))$. The percentages of the samples that fall within the ellipses are shown in each panel. The gray curves represent the trajectory by the rate equation starting from the initial point marked by the blue circles. The system parameters are $k_1 = 0.5$, $k_2 = 1.5$, $k_3 = 1.0$, $k_4 = 1.0$, and $\Omega = 10^4$. The initial point is $(x_0^*, y_0^*) = (0.8, 2.6)$.

remembrance of things past

noisy dynamics of a nonlinear system is fundamentally different from Brownian motion, as the flow ALWAYS induces a local, history dependent effective noise

local Fokker-Planck operator

let

$$\{\ldots, x_{-1}, x_0, x_1, x_2, \ldots\}$$

be a deterministic trajectory

$$x_{a+1} = f(x_a)$$

noisy trajectory is centered on the deterministic trajectory

$$x = x_a + z_a\,, \qquad f_a(z_a) = f(x_a + z_a) - x_{a+1}$$

local Fokker-Planck operator

$$\mathcal{L}_{FPa}(z_{a+1}, z_a) = \frac{1}{\sqrt{4\pi D}} \exp\left[-\frac{(z_{a+1} - f_a(z_a))^2}{4D}\right]$$

Fokker-Planck formulation replaces individual noisy trajectories by evolution of their densities

$$\mathcal{L}_{FP}^{k}(z_k, z_0) \,=\, \int [dz]\, e^{-\frac{1}{2}\sum_a (z_{a+1}-f_a(z_a))^T \frac{1}{\Delta}(z_{a+1}-f_a(z_a))}$$

evolution to time k is given by the d-dimensional path integral over the $k-1$ intermediate noisy trajectory points

$$\mathcal{L}^{k}_{FP}(z_k, z_0) \,=\, \int [dz]\, e^{-\frac{1}{2}\sum_a (z_{a+1}-f_a(z_a))^T \frac{1}{\Delta}(z_{a+1}-f_a(z_a))}$$

zero mean; covariance matrix / diffusion tensor Δ

$$\langle \xi_j(t_a) \rangle = 0\,, \qquad \langle \xi_{a,i}\, \xi^T_{a,j} \rangle = \Delta_{ij}\,,$$

where $\langle \cdots \rangle$ stands for ensemble average over many realizations of the noise map $f(x_a)$ is nonlinear. Taylor expand

$$f_a(z_a) = M_a z_a + \cdots$$

approximate the noisy map by its linearized action,

$$z_{a+1} = M_a z_a + \xi_a \,,$$

where M_a is the Jacobian matrix, $(M_a)_{ij} = \partial f(x_a)_i / \partial x_j$

M_a is the Jacobian matrix, $(M_a)_{ij} = \partial f(x_a)_i / \partial x_j$

linearized Fokker-Planck operator

$$\mathcal{L}_{FPa}(z_{a+1}, z_a) = \frac{1}{N}\, e^{-\frac{1}{2}(z_{a+1}-M_a z_a)^T \frac{1}{\Delta}(z_{a+1}-M_a z_a)}$$

[Kalman filter ‘prediction’, WKB, semiclassical, saddlepoint, ... approximation] linearized evolution operator maps a cigar-shaped Gaussian density distribution with covariance matrix Q_a at time a

$$\rho_a(z_a) = \frac{1}{C_a} e^{-\frac{1}{2} z_a^T \frac{1}{Q_a} z_a}$$

into cigar

$$\rho_{a+1}(z_{a+1}) = \int dz_a \, \mathcal{L}_{FP a}(z_{a+1}, z_a) \, \rho_a(z_a)$$

one time step later

rolled your own cigar

convolution of a Gaussian with a Gaussian is again a Gaussian. Integrate, obtain that

the covariance of the transported packet is given by

evolution law for the covariance matrix Q_a

$$Q_{a+1} = M_a Q_a M_a^T + \Delta_a$$

rolled your own cigar

evolution law for the covariance matrix Q_a

$$Q_{a+1} = M_a Q_a M_a^T + \Delta_a$$

in one time step a Gaussian density distribution with covariance matrix Q_a is smeared into a Gaussian 'cigar' whose widths and orientation are given by eigenvalues and eigenvectors of Q_{a+1}

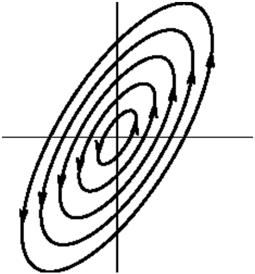
(1) deterministically transported and deformed local density covariance matrix $Q \to MQM^T$, and

(2) and noise covariance matrix Δ

add up as sums of squares

everybody understands how to fall into a hole :)

perturbations are
bounded



oscillatory eigenmodes,
crystals,
solid state physics

digressions die out

solitary death is local

or on life support

chaos

what is chaos ?

chaos is

"sensitivity to initial conditions" !!!

what is chaos ?

chaos is

"sensitivity to initial conditions" !!!

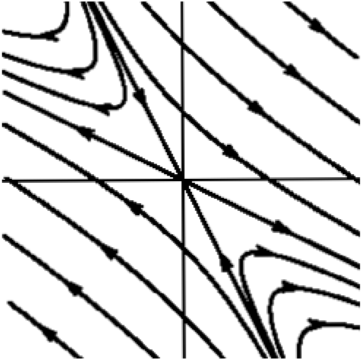
nu ?

what does that mean ???

computers damage brains, not optimally ...

what is chaos ?

is it local instability ?



is hyperbolicity chaos, turbulence ?

what is chaos ?

Poincaré says

chaos is

the ∞ of global recurrences

what is chaos ?

Poincaré says

chaos is

the ∞ of global recurrences

a catalogue of global recurrences

chaos is the Book of Lives obeying The Law turbulence.zip

or ‘equation assisted’ data compression:

replace the ∞ of turbulent videos by the best possible

small finite set

of videos encoding all physically distinct motions of the turbulent fluid

dynamical system

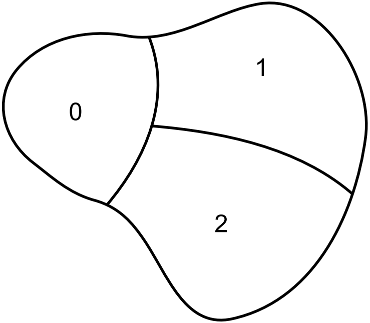
the pair $(\mathcal{M}, f)$

the problem

enumerate, classify all solutions of $(\mathcal{M}, f)$

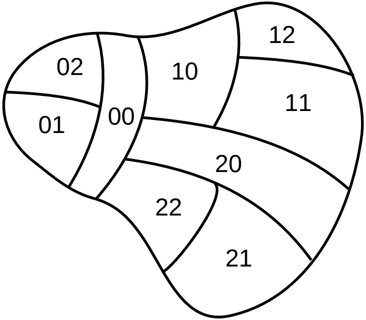
deterministic partition into regions of similar states

1-step memory partition



$\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2$
ternary alphabet
$\mathcal{A} = \{1, 2, 3\}$.

2-step memory refinement



$\mathcal{M}_i = \mathcal{M}_{i0} \cup \mathcal{M}_{i1} \cup \mathcal{M}_{i2}$
labeled by nine ‘words’
$\{00, 01, 02, \cdots, 21, 22\}$.

deterministic partitions are no good

deterministic dynamics: partitioning can be arbitrarily fine

requires exponential # of exponentially small regions

deterministic partitions are no good

deterministic dynamics: partitioning can be arbitrarily fine

requires exponential # of exponentially small regions

yet

in practice

every physical problem must be coarse partitioned

noise rules the state space

in practice

every neuronal problem is coarse partitioned by noise

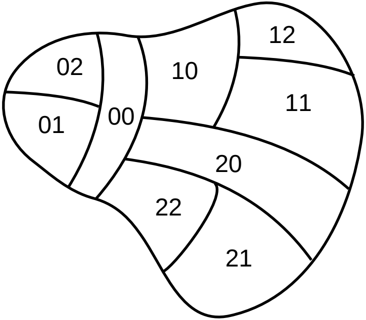
noise rules the state space

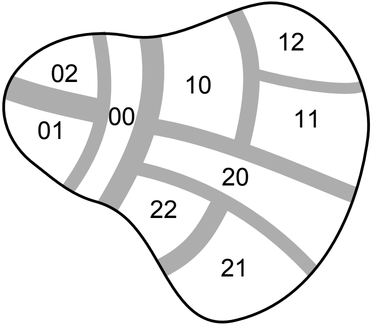
in practice

every neuronal problem is coarse partitioned by noise

- any neuronal system experiences intrinsic noise
- any model computation is ‘noisy’
- any prediction only needs a desired finite accuracy

deterministic vs. noisy partitions





deterministic partition

noise blurs the boundaries

can be refined
ad infinitum

when overlapping, no further refinement of partition

deterministic, idealized state space

a manifold $\mathcal{M} \in \mathbb{R}^d$: d real numbers determine the state of the system $x \in \mathcal{M}$

noise-limited state space

a 'grid' $\mathcal{M}'$: N discrete states of the system $a \in \mathcal{M}'$, one for each noise covariance ellipsoid Δ_a

dynamics + noise: unique coarse-grained partition

reasonable to assume that the noise

limits the resolution
that can be attained in partitioning the state space

dynamics + noise: unique coarse-grained partition

reasonable to assume that the noise

is uniform,
leading to a uniform grid partition of the state space

dynamics + noise: unique coarse-grained partition

reasonable to assume that the noise

is uniform,
leading to a uniform grid partition of the state space

in dynamics, this is wrong!

noise has memory

dynamics + noise: unique coarse-grained partition

noise memory

accumulated noise along dynamical trajectories
always coarsens the partition nonuniformly

dynamics + noise: unique coarse-grained partition

noise memory

accumulated noise along dynamical trajectories
always coarsens the partition nonuniformly

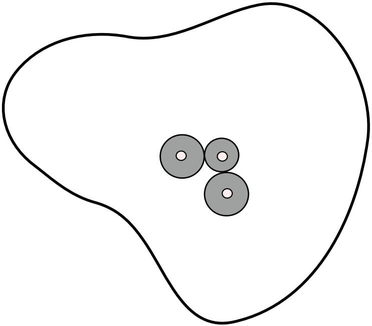
that is good, because

dynamics + noise determine

the **finest attainable** partition

noise limited state space partitions

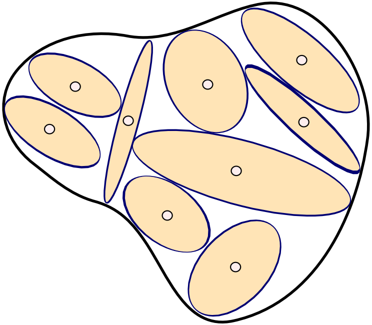
noise limited cell



a resolvable neighborhood is no smaller than a ball whose radius is the noise amplitude

this is wrong!

noise limited partition grid



state space noise-partitioned into neighborhoods indicated by their centers

periodic points instead of boundaries

- mhm, do not know how to compute boundaries...
- however, each partition contains a short periodic point smeared into a ‘cigar’ by noise

periodic points instead of boundaries

- each partition contains a short periodic point smeared into a ‘cigar’ by noise

compute the size of a noisy periodic point neighborhood!

things fall apart, centre cannot hold

remember :

if all eigenvalues of M are strictly contracting

covariance evolution

$$Q_{n+1} = M_n Q_n M_n^T + \Delta_n \,, \quad |\Lambda_j| < 1$$

drives any measure to the unique invariant Gaussian measure

Q. what if M has expanding eigenvalues?

both deterministic dynamics and noise tend to smear densities away from the fixed point: no peaked Gaussian in your future

things fall apart, centre cannot hold

remember :

if all eigenvalues of M are strictly contracting

covariance evolution

$$Q_{n+1} = M_n Q_n M_n^T + \Delta_n \,, \quad |\Lambda_j| < 1$$

drives any measure to the unique invariant Gaussian measure

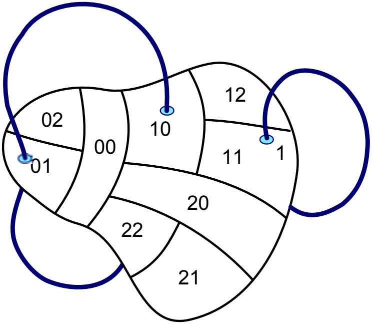
Q. what if M has expanding eigenvalues?

A. look into the past, for initial peaked distribution that spreads to the present state

$$\tilde{Q}_{n+1} + \Delta_n = M_n \tilde{Q}_n M_n^T \,, \quad |\Lambda_j| > 1 \,,$$

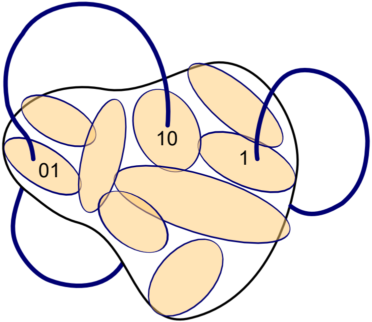
periodic orbit partition

deterministic partition



some short periodic points:
fixed point $\overline{1} = \{x_1\}$
two-cycle $\overline{01} = \{x_{01}, x_{10}\}$

noisy partition



periodic points blurred by the noise into cigar-shaped densities

strategy

- use periodic orbits to partition state space
- compute local eigenfunctions of the Fokker-Planck operator to determine their neighborhoods
- done once neighborhoods overlap

- successive refinements of a deterministic partition: exponentially shrinking neighborhoods
- as the periods of periodic orbits increase, the diffusion always wins:

partition stops at the finest attainable partition, beyond which the diffusive smearing exceeds the size of any deterministic subpartition.

- the local diffusion rate differs from a trajectory to a trajectory, as different neighborhoods merge at different times, so

there is *no one single time* beyond which noise takes over

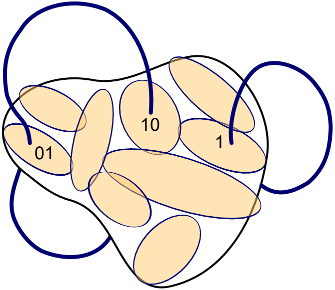
periodic points and their cigars

- each partition contains a short periodic point smeared into a ‘cigar’ by noise

periodic points and their cigars

- each partition contains a short periodic point smeared into a ‘cigar’ by noise
- compute the size of a noisy periodic point neighborhood!

noisy periodic orbit partition



optimal partition hypothesis

optimal partition:
the maximal set of resolvable periodic point neighborhoods

why care?

if the high-dimensional flow has only a few unstable directions, the overlapping stochastic ‘cigars’ provide a *compact cover* of the noisy chaotic attractor, embedded in a state space of arbitrarily high dimension

adjoint Fokker-Planck operator

to estimate the size of a noisy neighborhood of a trajectory point x_a along its *unstable* directions, we need to determine the effect of noise on the points *preceding* x_a

this is described by the *adjoint Fokker-Planck operator*

$$\begin{aligned}
\tilde{\rho}(y, k-1) &= \mathcal{L}^{\dagger}_{FP} \circ \tilde{\rho}(y, k) \\
&= \int [dy]\, \exp\left\{-\frac{1}{2}(y - f(x))^T \frac{1}{\Delta}(y - f(x))\right\} \tilde{\rho}(y, k)\,,
\end{aligned}$$

carries a density concentrated around the previous point x_{n-1} to a density concentrated around x_n

things fall apart, centre cannot hold

but what if M has *expanding* eigenvalues?

both deterministic dynamics and noise tend to smear densities away from the fixed point: no peaked Gaussian in your future

things fall apart, centre cannot hold

but what if M has *expanding* eigenvalues?

look into the past, for initial peaked distribution that spreads to the present state

for unstable directions, look back

if M has only *expanding* eigenvalues,

balance between the two is attained by iteration from the past, and the evolution of the covariance matrix $\tilde{Q}$ is now given by

$$\tilde{Q}_{n+1} + \Delta_n = M_n \tilde{Q}_n M_n^T \,,$$

[aside to control theorists: reachability and observability Gramians]

solving the Lyapunov equation

iterate $Q_{n+1} = M_n Q_n M_n^T + \Delta_n$
attractive fixed point, $Q = Q_\infty$, $M = M_n$, $Q = Q_n$:

$$Q \;=\; \Delta + M\Delta M^\top + M^2\Delta(M^\top)^2 + \cdots = \sum_{m,n=0}^{\infty} \delta_{mn} M^n \Delta (M^\top)^m$$

bring to resolvent form, $\delta_{mn} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta(m-n)}$

for M contracting, expanding, or hyperbolic (!)

$$Q = \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{1}{\mathbf{1} - e^{-i\theta} M} \Delta \frac{1}{\mathbf{1} - e^{i\theta} M^\top}$$

Cauchy magic

a similarity transformation S separates the expanding and contracting subspaces

$$\Lambda \equiv S^{-1}MS = \left[\begin{array}{cc} \Lambda_e & 0 \\ 0 & \Lambda_c \end{array}\right]$$

transformed noise covariance matrix

$$\hat{\Delta} \equiv S^{-1}\Delta(S^{-1})^{\top} = \left[\begin{array}{cc} \Delta_{ee} & \Delta_{ec} \\ \Delta_{ce} & \Delta_{cc} \end{array}\right]$$

Cauchy magic

contour integral representation

$$Q = \oint_{\Gamma} \frac{ds}{2\pi} (\mathbf{1} - s^{-1} M)^{-1} \Delta (\mathbf{1} - sM)^{-1}$$

separates Q into expanding and contracting covariances:

$$\tilde{Q}_e \equiv S \begin{bmatrix} Q_e & 0 \\ 0 & 0 \end{bmatrix} S^{\top}, \quad Q_c \equiv S \begin{bmatrix} 0 & 0 \\ 0 & Q_c \end{bmatrix} S^{\top}$$

two stationary ‘cigars’, one in the expanding manifold and the other in the contracting manifold (not orthogonal to each other!)

optimal partition challenge

finally in position to address our challenge:

determine the finest possible partition for a given noise

local problem solved: can compute every cigar

a periodic point of period *n* is a fixed point of *n*th iterate of dynamics

global problem solved: can compute all cigars

more algebra: can compute the noisy neighborhoods of all periodic points

does it work?

evaluation of these Gaussian densities requires no Fokker-Planck PDE formalism

width of a Gaussian packet centered on a trajectory is fully specified by a deterministic computation that is already a pre-computed byproduct of the periodic orbit computations: the deterministic orbit and its linear stability

resolution of a one-dimensional chaotic repeller

As an illustration of the method, consider the chaotic repeller on the unit interval

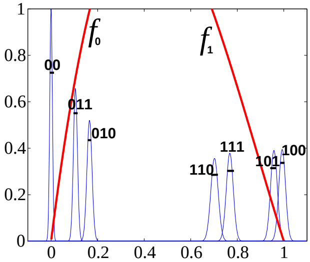
$$x_{n+1} = \Lambda_0 \, x_n (1 - x_n)(1 - b x_n) + \xi_n \,, \quad \Lambda_0 = 8,\ b = 0.6 \,,$$

with noise strength $2D = 0.002$

optimal partition, 1 dimensional map

f_0, f_1: branches of deterministic map

a deterministic orbit itinerary is given by the $\{f_0, f_1\}$ branches visitation sequence



[symbolic dynamics, however, is not a prerequisite for implementing the method]

'the best possible of all partitions' hypothesis formulated as an algorithm

- calculate the local adjoint Fokker-Planck operator eigenfunction width Q_a for every unstable periodic point x_a
- assign one-standard deviation neighborhood $[x_a - Q_a, x_a + Q_a]$ to every unstable periodic point x_a
- cover the state space with neighborhoods of orbit points of higher and higher period n_p
- stop refining the local resolution whenever the adjacent neighborhoods of x_a and x_b overlap:

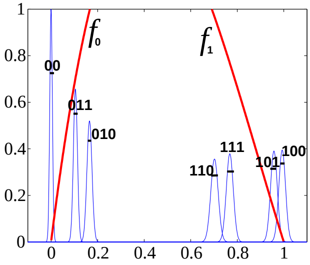
$$|x_a - x_b| < Q_a + Q_b$$

optimal partition, 1 dimensional map

f_0, f_1: branches of deterministic map

local eigenfunctions $\tilde{\rho}_a$ partition state space by neighborhoods of periodic points of period 3

neighborhoods $\mathcal{M}_{000}$ and $\mathcal{M}_{001}$ overlap, so $\mathcal{M}_{00}$ cannot be resolved further



all neighborhoods $\{\mathcal{M}_{0101}, \mathcal{M}_{0100}, \cdots\}$ of period $n_p = 4$ cycle points overlap, so

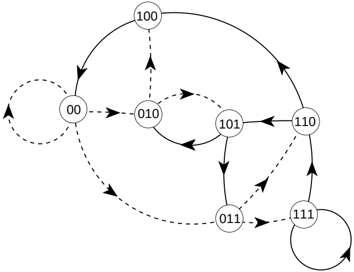
state space can be resolved into 7 neighborhoods

$$\{\mathcal{M}_{00}, \mathcal{M}_{011}, \mathcal{M}_{010}, \mathcal{M}_{110}, \mathcal{M}_{111}, \mathcal{M}_{101}, \mathcal{M}_{100}\}$$

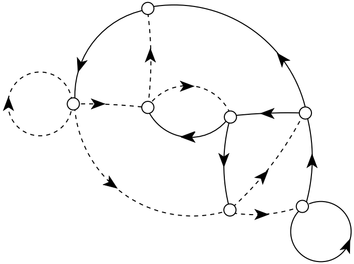
Markov partition

evolution in time maps intervals
$\mathcal{M}_{011} \to \{\mathcal{M}_{110}, \mathcal{M}_{111}\}$
$\mathcal{M}_{00} \to \{\mathcal{M}_{00}, \mathcal{M}_{011}, \mathcal{M}_{010}\}$, etc..

summarized by the transition graph (links correspond to elements of transition matrix T_{ba}): the regions b that can be reached from the region a in one time step



transition graph



7 nodes = 7 regions of the optimal partition

dotted links = symbol 0 (next region reached by f_0)

full links = symbol 1 (next region reached by f_1)

region labels in the nodes can be omitted, with links keeping track of the symbolic dynamics

(1) deterministic dynamics is full binary shift, but

(2) noise dynamics nontrivial and *finite*

predictions

escape rate and the Lyapunov exponent of the repeller

are given by the leading eigenvalue of this $[7 \times 7]$ graph / transition matrix

tests : numerical results are consistent with the full Fokker-Planck PDE simulations

what is novel?

- we have shown how to compute the **locally optimal partition**, for a given dynamical system and given noise, in terms of local eigenfunctions of the forward-backward actions of the Fokker-Planck operator and its adjoint

what is novel?

- **A handsome reward:** as the optimal partition is always finite, the dynamics on this ‘best possible of all partitions’ is encoded by a finite transition graph of finite memory, and the Fokker-Planck operator can be represented by a finite matrix

the payback

claim:

optimal partition hypothesis

- the best of all possible state space partitions
- optimal for the given noise

the payback

claim:

optimal partition hypothesis

- optimal partition replaces stochastic PDEs by finite, low-dimensional Fokker-Planck matrices

the payback

claim:

optimal partition hypothesis

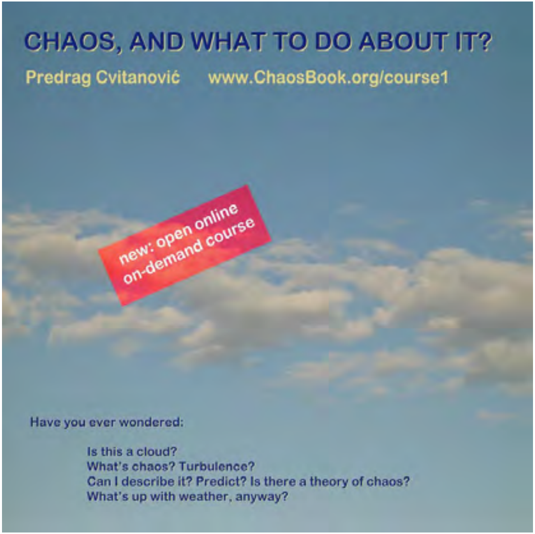
- optimal partition replaces stochastic PDEs by finite, low-dimensional Fokker-Planck matrices
- finite matrix calculations, finite cycle expansions $\Rightarrow$ optimal estimates of long-time observables (escape rates, Lyapunov exponents, etc.)

noise is your friend

references

- D. Lippolis and P. Cvitanović, *How well can one resolve the state space of a chaotic map?*, Phys. Rev. Lett. 104, 014101 (2010); arXiv.org:0902.4269
- P. Cvitanović and D. Lippolis, *Knowing when to stop: How noise frees us from determinism*, in M. Robnik and V.G. Romanovski, eds., *Let's Face Chaos through Nonlinear Dynamics* (Am. Inst. of Phys., Melville, New York, 2012); arXiv.org:1206.5506
- J. M. Heninger, D. Lippolis and P. Cvitanović, *Neighborhoods of periodic orbits and the stationary distribution of a noisy chaotic system* Phys. Rev. E 92, 062922 (2015); arXiv.org:1507.00462

what next? take the course!



student raves :

...10^6 times harder than any other online course...

the rest is noise

brief history of noise

literature on stochastic dynamical systems is vast, starts with the Laplace 1810 memoir

all of this literature assumes uniform / bounded hyperbolicity and seeks to define a single, globally averaged diffusion induced average resolution (Heisenberg time, in the context of semi-classical quantization).

appears to have been first introduced by Wiener as the exact solution for a purely diffusive Wiener-Lévy process in one dimension.
Onsager and Machlup use it in their variational principle to study thermodynamic fluctuations in a neighborhood of single, linearly attractive equilibrium point (i.e., without any dynamics).

dynamical 'action' Lagrangian, and symplectic noise Hamiltonian were first written down by Freidlin and Wentzell (1970's), whose formulation of the 'large deviation principle' was inspired by the Feynman quantum path integral (1940's). Feynman, in turn, followed Dirac (1933's) who was the first to discover that in the short-time limit the quantum propagator (imaginary time, quantum sibling of the Wiener stochastic distribution) is exact. Gaspard: 'pseudo-energy of the Onsager-Machlup-Freidlin-Wentzell scheme.' Roncadelli: the 'Wiener-Onsager-Machlup Lagrangian.'

noisy flow

here we briefly repeat the derivation of local Fokker-Planck operator for a continuous time flow

d-dimensional stochastic flow

$$\frac{dx}{dt} = v(x) + \hat{\xi}(t)\,,$$

deterministic velocity field $v(x)$, called 'drift' in the stochastic literature

density evolution

in time $\delta\tau$ the deterministic trajectory advances by $v(x_n)\,\delta\tau$.

the probability that the trajectory reaches x_{n+1}

$$\mathcal{L}^{\delta\tau}_{FP}(x_{n+1}, x_n) \;=\; \frac{1}{N}\exp\left[-\frac{1}{2\,\delta\tau}(\xi_n^T\frac{1}{\Delta}\xi_n)\right]\,.$$

density evolution

in time $\delta\tau$ the deterministic trajectory advances by $v(x_n)\,\delta\tau$.

the probability that the trajectory reaches x_{n+1}

$$\mathcal{L}_{FP}^{\delta\tau}(x_{n+1}, x_n) \;=\; \frac{1}{N}\exp\left[-\frac{1}{2\,\delta\tau}(\xi_n^T\frac{1}{\Delta}\xi_n)\right] \,.$$

ξ_n is the deviation of the noisy trajectory from the deterministic one,

$$\xi_n = \delta x_n - v(x_n)\,\delta\tau \,,$$

the probability that the trajectory reaches x_{n+1}

$$\mathcal{L}^{\delta\tau}_{FP}(x_{n+1}, x_n) \;=\; \frac{1}{N} \exp\left[-\frac{1}{2\,\delta\tau} (\xi_n^T \frac{1}{\Delta} \xi_n) \right] \, .$$

ξ_n is the deviation of the noisy trajectory from the deterministic one,

$$\xi_n = \delta x_n - v(x_n)\,\delta\tau \, ,$$

$$\delta x_n = x_{n+1} - x_n \simeq \dot{x}_n \,\delta\tau \, , \qquad f^{\delta\tau}(x_n) - x_n \simeq v(x_n)\,\delta\tau \, ,$$

density evolution

the probability that the trajectory reaches x_{n+1}

$$\mathcal{L}_{FP}^{\delta\tau}(x_{n+1}, x_n) \;=\; \frac{1}{N} \exp\left[-\frac{1}{2\,\delta\tau} (\xi_n^T \frac{1}{\Delta} \xi_n) \right] .$$

where

$$\{x_0, x_1, \cdots, x_n, \cdots, x_k\} = \{x(0), x(\delta\tau), \cdots, x(n\delta\tau), \cdots, x(t)\}$$

is a sequence of $k+1$ points $x_n = x(t_n)$ along the noisy trajectory, separated by time increments $\delta\tau = t/k$

density evolution

finite time Fokker-Planck evolution $\rho(x,t) = \mathcal{L}^t_{FP} \circ \rho(x,0)$ of an initial density $\rho(x_0,0)$ is obtained by a sequence of consecutive short-time steps

$$\mathcal{L}^t_{FP}(x_k, x_0) = \int [dx] \, \exp\left\{ -\frac{1}{4D\delta\tau} \sum_{n=1}^{k-1} [x_{n+1} - f^{\delta\tau}(x_n)]^2 \right\} ,$$

(Gaussian) probability distribution function,

$$\mathcal{L}^t_{FP}(x, x_0) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left[-\frac{(x - x_0)^2}{2\sigma^2 t}\right]$$

variance $\sigma^2 t = 2Dt$, standard deviation $\sqrt{2Dt}$
uncorrelated in time

$$\langle x_{n+1} - x_n \rangle = 0\,, \qquad \langle (x_{m+1} - x_m)(x_{n+1} - x_n) \rangle = 2\,D\,\delta_{mn}$$

density evolution

in time $\delta\tau$ the deterministic trajectory advances by $v(x_n)\,\delta\tau$.

the probability that the trajectory reaches x_{n+1}

$$\mathcal{L}_{FP}^{\delta\tau}(x_{n+1}, x_n) \;=\; \frac{1}{N}\exp\left[-\frac{1}{2\,\delta\tau}(\xi_n^T\frac{1}{\Delta}\xi_n)\right]\,.$$

density evolution

in time $\delta\tau$ the deterministic trajectory advances by $v(x_n)\,\delta\tau$.

the probability that the trajectory reaches x_{n+1}

$$\mathcal{L}_{FP}^{\delta\tau}(x_{n+1}, x_n) \;=\; \frac{1}{N}\exp\left[-\frac{1}{2\,\delta\tau}(\xi_n^T\frac{1}{\Delta}\xi_n)\right] \,.$$

ξ_n is the deviation of the noisy trajectory from the deterministic one,

$$\xi_n = \delta x_n - v(x_n)\,\delta\tau \,,$$

density evolution

the probability that the trajectory reaches x_{n+1}

$$\mathcal{L}^{\delta\tau}_{FP}(x_{n+1}, x_n) \;=\; \frac{1}{N}\exp\left[-\frac{1}{2\,\delta\tau}(\xi_n^T\frac{1}{\Delta}\xi_n)\right] \;.$$

ξ_n is the deviation of the noisy trajectory from the deterministic one,

$$\xi_n = \delta x_n - v(x_n)\,\delta\tau \;,$$

$$\delta x_n = x_{n+1} - x_n \simeq \dot{x}_n\,\delta\tau \;, \qquad f^{\delta\tau}(x_n) - x_n \simeq v(x_n)\,\delta\tau \;,$$

density evolution

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where

$$\{x_0, x_1, \cdots, x_n, \cdots, x_k\} = \{x(0), x(\delta\tau), \cdots, x(n\delta\tau), \cdots, x(t)\}$$

is a sequence of $k+1$ points $x_n = x(t_n)$ along the noisy trajectory, separated by time increments $\delta\tau = t/k$

zero mean and covariance matrix (diffusion tensor)

$$\langle \xi_j(t_n) \rangle = 0 \,, \qquad \langle \xi_i(t_m)\, \xi_j^T(t_n) \rangle = \Delta_{ij}\, \delta_{nm} \,,$$

where $\langle \cdots \rangle$ stands for ensemble average over many realizations of the noise.

density evolution

Fokker-Planck formulation replaces individual noisy trajectories by the evolution of their density
finite time Fokker-Planck evolution $\rho(x,t) = \mathcal{L}^t_{FP} \circ \rho(x,0)$ of an initial density $\rho(x_0,0)$ is obtained by a sequence of consecutive short-time steps

$$\mathcal{L}^t_{FP}(x_k, x_0) = \int [dx] \, \exp\left\{ -\frac{1}{4D\delta\tau} \sum_{n=1}^{k-1} [x_{n+1} - f^{\delta\tau}(x_n)]^2 \right\} \,,$$

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as a stochastic path (Wiener) integral

associated continuous time Fokker-Planck equation for the time evolution of a density of noisy trajectories is

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predictions

- finite partition $\Rightarrow$ finite Fokker-Planck matrix
- its determinant yields time averages of dynamical observables

questions?

- how to combine Fokker-Planck and adjoint Fokker-Planck operators to describe hyperbolic periodic points (saddles)?

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- how to combine Fokker-Planck and adjoint Fokker-Planck operators to describe hyperbolic periodic points (saddles)? Hint: like H. H. Rugh (1992)?

  He combined deterministic evolution operator and adjoint operators to describe hyperbolic periodic points (saddles)

questions?

- apply to Navier-Stokes turbulence?

  computation of unstable periodic orbits in high-dimensional state spaces, such as Navier-Stokes, is at the border of what is feasible numerically, and criteria to identify finite sets of the most important solutions are very much needed. Where are we to stop calculating orbits of a given hyperbolic flow?

summary

- Computation of unstable periodic orbits in high-dimensional state spaces, such as Navier-Stokes, is at the border of what is feasible numerically, and criteria to identify finite sets of the most important solutions are very much needed. Where are we to stop calculating orbits of a given hyperbolic flow?

- Intuitively, as we look at longer and longer periodic orbits, their neighborhoods shrink exponentially with time, while the variance of the noise-induced orbit smearing remains bounded; there has to be a *turnover time*, a time at which the noise-induced width overwhelms the exponentially shrinking deterministic dynamics, so that no better resolution is possible. Given a specified (possibly state space dependent) noise, we need to find, periodic orbit by periodic orbit, whether a further sub-partitioning is possible.

summary

- We have described here the *optimal partition hypothesis*, a new method for partitioning the state space of a chaotic repeller in presence of weak Gaussian noise, and tested the method in a 1-dimensional setting against direct numerical Fokker-Planck operator calculation.