Chapter 6

Lyapunov exponents

> [...] people should be taught linear algebra a lot earlier than they are now, because it short-circuits a lot of really stupid and painful and idiotic material.
> — Stephen Boyd

Let us apply our newly acquired tools to the fundamental diagnostics in dynamics: Is a given system 'chaotic'? And if so, how chaotic? If all points in a neighborhood of a trajectory converge toward the same orbit, the attractor is a fixed point or a limit cycle. However, if the attractor is strange, any two trajectories $x(t) = f^t(x_0)$ and $x(t) + \delta x(t) = f^t(x_0 + \delta x_0)$ that start out very close to each other separate exponentially with time, and in a finite time their separation attains the size of the accessible state space.

example 2.3
section 1.3.1
remark 6.1

This *sensitivity to initial conditions* can be quantified as

$$\|\,\delta x(t)\,\| \approx e^{\lambda t}\,\|\,\delta x_0\,\| \tag{6.1}$$

where λ, the mean rate of separation of trajectories of the system, is called the leading *Lyapunov exponent*. In the limit of infinite time the Lyapunov exponent is a global measure of the rate at which nearby trajectories diverge, averaged over the strange attractor. As it so often goes with easy ideas, it turns out that Lyapunov exponents are not natural for study of dynamics, and we would have passed them over in silence, were it not for so much literature that talks about them. So in a textbook we are duty bound to explain what all the excitement is about. But then we round the chapter off with a scholarly remark almost as long as the chapter itself: we do not recommend that you evaluate Lyapunov exponents and Lyapunov singular vectors. Compute the stability exponents / covariant vectors.

6.1 Stretch, strain and twirl

> Diagonalizing the matrix: that's the key to the whole thing.
> — Governor Arnold Schwarzenegger

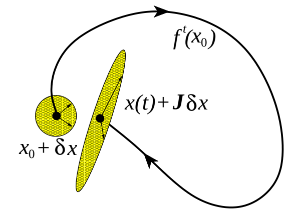

Figure 6.1: The linearized flow maps a swarm of initial points in an infinitesimal spherical neighborhood of squared radius δx^2 at x_0 into an ellipsoid $\delta x^\top (J^\top J)\,\delta x$ at $x(t)$ a finite time t later, rotated and stretched/compressed along the principal axes by streches $\{\sigma_j\}$.

In general the Jacobian matrix J is neither diagonal, nor diagonalizable, nor constant along the trajectory. What is a geometrical meaning of the mapping of a neighborhood by J? Here the continuum mechanics insights are helpful, in particular the polar decomposition which affords a visualization of the linearization of a flow as a mapping of the initial ball into an ellipsoid (figure 6.1).

First, a few definitions: A symmetric $[d \times d]$ matrix Q is *positive definite*, $Q > 0$, if $x^\top Q x > 0$ for any nonzero vector $x \in \mathbb{R}^d$. Q is *negative definite*, $Q < 0$, if $x^\top Q x < 0$ for any nonzero vector x. Alternatively, Q is a positive (negative) definite matrix if all its eigenvalues are positive (negative). A matrix R is orthogonal if $R^\top R = \mathbf{1}$, and proper orthogonal if $\det R = +1$. Here the superscript $^\top$ denotes the transpose. For example, $(x_1, \cdots, x_d)$ is a row vector, $(x_1, \cdots, x_d)^\top$ is a column vector.

By the polar decomposition theorem, a deformation J can be factored into a rotation R and a right / left stretch tensor U / V, remark 6.2

$$J = RU = VR\,, \tag{6.2}$$

where R is a proper-orthogonal matrix and U, V are symmetric positive definite matrices with strictly positive real eigenvalues $\{\sigma_1, \sigma_2, \cdots, \sigma_d\}$ called *principal stretches* (singular values, Hankel singular values), and with orthonormal eigenvector bases,

$$\begin{aligned} U\,u^{(i)} &= \sigma_i u^{(i)}\,, \qquad \{u^{(1)}, u^{(2)}, \cdots, u^{(d)}\} \\ V\,v^{(i)} &= \sigma_i v^{(i)}\,, \qquad \{v^{(1)}, v^{(2)}, \cdots, v^{(d)}\}\,. \end{aligned} \tag{6.3}$$

$\sigma_i > 1$ for stretching and $0 < \sigma_i < 1$ for compression along the direction $u^{(i)}$ or $v^{(i)}$. $\{u^{(j)}\}$ are the *principal axes of strain* at the initial point x_0; $\{v^{(j)}\}$ are the principal axes of strain at the present placement x. From a geometric point of view, J maps the unit sphere into an ellipsoid, figure 6.1; the principal stretches are then the lengths of the semiaxes of this ellipsoid. The rotation matrix R carries the initial axes of strain into the present ones, $V = RUR^\top$. The eigenvalues of the

remark 6.1

$$\begin{aligned} \text{right Cauchy-Green strain tensor:} &\quad J^\top J = U^2 \\ \text{left Cauchy-Green strain tensor:} &\quad J\,J^\top = V^2 \end{aligned} \tag{6.4}$$

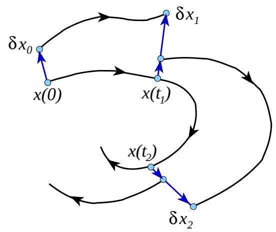

Figure 6.2: A long-time numerical calculation of the leading Lyapunov exponent requires rescaling the distance in order to keep the nearby trajectory separation within the linearized flow range.

are $\{\sigma_j^2\}$, the squares of principal stretches.

example 6.2
p. 123

6.2 Lyapunov exponents

(J. Mathiesen and P. Cvitanović)

The mean growth rate of the distance $\|\delta x(t)\| / \|\delta x_0\|$ between neighboring trajectories (6.1) is given by the leading *Lyapunov exponent* which can be estimated for long (but not too long) time t as

$$\lambda \simeq \frac{1}{t} \ln \frac{\|\delta x(t)\|}{\|\delta x(0)\|} \tag{6.5}$$

For notational brevity we shall often suppress the dependence of quantities such as $\lambda = \lambda(x_0, t)$, $\delta x(t) = \delta x(x_0, t)$ on the initial point x_0. One can use (6.5) as is, take a small initial separation δx_0, track the distance between two nearby trajectories until $\|\delta x(t_1)\|$ gets significantly big, then record $t_1 \lambda_1 = \ln(\|\delta x(t_1)\| / \|\delta x_0\|)$, rescale $\delta x(t_1)$ by factor $\delta x_0 / \delta x(t_1)$, and continue add infinitum, as in figure 6.2, with the leading Lyapunov exponent given by

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \sum_i t_i \lambda_i \,, \qquad t = \sum_i t_i \,. \tag{6.6}$$

Deciding what is a safe 'linear range', the distance beyond which the separation vector $\delta x(t)$ should be rescaled, is a dark art.

We can start out with a small δx and try to estimate the leading Lyapunov exponent λ from (6.6), but now that we have quantified the notion of linear stability in chapter 4, we can do better. The problem with measuring the growth rate of the distance between two points is that as the points separate, the measurement is less and less a local measurement. In the study of experimental time series this might be the only option, but if we have equations of motion, a better way is to measure the growth rate of vectors transverse to a given orbit.

Given the equations of motion, for infinitesimal δx we know the $\delta x_i(t)/\delta x_j(0)$ ratio exactly, as this is by definition the Jacobian matrix

$$\lim_{\delta x(0)\to 0} \frac{\delta x_i(t)}{\delta x_j(0)} = \frac{\partial x_i(t)}{\partial x_j(0)} = J^t_{ij}(x_0)\,,$$

so the leading Lyapunov exponent can be computed from the linearization (4.16)

$$\lambda(x_0) = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\left\| J^t(x_0)\, \delta x_0 \right\|}{\|\delta x_0\|} = \lim_{t\to\infty} \frac{1}{2t} \ln\left(\hat{n}^\top J^{t\top} J^t \hat{n}\right) . \tag{6.7}$$

In this formula the scale of the initial separation drops out, only its orientation given by the initial orientation unit vector $\hat{n} = \delta x_0 / \|\delta x_0\|$ matters. If one does not care about the orientation of the separation vector between a trajectory and its perturbation, but only its magnitude, one can interpret $\left\| J^t \delta x_0 \right\|^2 = \delta x_0{}^\top (J^{t\top} J^t)\, \delta x_0$, as the *error correlation matrix*. In the continuum mechanics language, the right Cauchy-Green strain tensor $J^\top J$ (6.4) is the natural object to describe how linearized neighborhoods deform. In the theory of dynamical systems the *stretches* of continuum mechanics are called the *finite-time Lyapunov* or *characteristic* exponents,

$$\lambda(x_0, \hat{n}; t) = \frac{1}{t} \ln \left\| J^t \hat{n} \right\| = \frac{1}{2t} \ln\left(\hat{n}^\top J^{t\top} J^t \hat{n}\right) . \tag{6.8}$$

They depend on the initial point x_0 and on the direction of the unit vector $\hat{n}$, $\|\hat{n}\| = 1$ at the initial time. If this vector is aligned along the ith principal stretch, $\hat{n} = u^{(i)}$, then the corresponding finite-time Lyapunov exponent (rate of stretching) is given by

$$\lambda_j(x_0; t) = \lambda(x_0, u^{(j)}; t) = \frac{1}{t} \ln \sigma_j(x_0; t). \tag{6.9}$$

We do not need to compute the strain tensor eigenbasis to determine the leading *Lyapunov exponent*,

$$\lambda(x_0, \hat{n}) = \lim_{t\to\infty} \frac{1}{t} \ln \left\| J^t \hat{n} \right\| = \lim_{t\to\infty} \frac{1}{2t} \ln\left(\hat{n}^\top J^{t\top} J^t \hat{n}\right) , \tag{6.10}$$

as expanding the initial orientation in the strain tensor eigenbasis (6.3), $\hat{n} = \sum(\hat{n} \cdot u^{(i)})u^{(i)}$, we have

$$\hat{n}^\top J^{t\top} J^t \hat{n} = \sum_{i=1}^{d} (\hat{n} \cdot u^{(i)})^2 \sigma_i^2 = (\hat{n} \cdot u^{(1)})^2 \sigma_1^2 \left(1 + O(\sigma_2^2/\sigma_1^2)\right) ,$$

with stretches ordered by decreasing magnitude, $\sigma_1 > \sigma_2 \geq \sigma_3 \cdots$. For long times the largest stretch dominates exponentially in (6.10), provided the orientation $\hat{n}$ of the initial separation was not chosen perpendicular to the dominant expanding eigen-direction $u^{(1)}$. Furthermore, for long times $J^t \hat{n}$ is dominated by the largest stability multiplier Λ_1, so the leading Lyapunov exponent is

$$\begin{aligned}
\lambda(x_0) &= \lim_{t\to\infty} \frac{1}{t} \left\{ \ln \left\| \hat{n} \cdot \mathbf{e}^{(1)} \right\| + \ln |\Lambda_1(x_0, t)| + O(e^{-2(\lambda_1 - \lambda_2)t}) \right\} \\
&= \lim_{t\to\infty} \frac{1}{t} \ln |\Lambda_1(x_0, t)| \,,
\end{aligned} \tag{6.11}$$

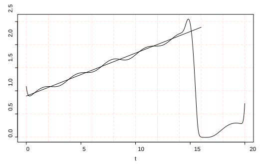

Figure 6.3: A numerical computation of the logarithm of the stretch $\hat{n}^\top(J^{t\top}J^t)\hat{n}$ in formula (6.10) for the Rössler flow (2.27), plotted as a function of the Rössler time units. The slope is the leading Lyapunov exponent $\lambda \approx 0.09$. The exponent is positive, so numerics lends credence to the hypothesis that the Rössler attractor is chaotic. The big unexplained jump illustrates perils of Lyapunov exponents numerics. (J. Mathiesen)

where $\Lambda_1(x_0, t)$ is the leading eigenvalue of $J^t(x_0)$. The leading Lyapunov exponent now follows from the Jacobian matrix by numerical integration of (4.10). The equations can be integrated accurately for a finite time, hence the infinite time limit of (6.7) can be only estimated from a finite set of evaluations of $\frac{1}{2}\ln(\hat{n}^\top J^{t\top}J^t\hat{n})$ as function of time, such as figure 6.3 for the Rössler flow (2.27).

As the local expansion and contraction rates vary along the flow, the temporal dependence exhibits small and large humps. The sudden fall to a low value in figure 6.3 is caused by a close passage to a folding point of the attractor, an illustration of why numerical evaluation of the Lyapunov exponents, and proving the very existence of a strange attractor is a difficult problem. The approximately monotone part of the curve you can use (at your own peril) to estimate the leading Lyapunov exponent by a straight line fit.

As we can already see, we are courting difficulties if we try to calculate the Lyapunov exponent by using the definition (6.11) directly. First of all, the state space is dense with atypical trajectories; for example, if x_0 happens to lie on a periodic orbit p, λ would be simply $\ln|\sigma_{p,1}|/T_p$, a local property of cycle p, not a global property of the dynamical system. Furthermore, even if x_0 happens to be a 'generic' state space point, it is still not obvious that $\ln|\sigma_{p,1}(x_0, t)|/t$ should be converging to anything in particular. In a Hamiltonian system with coexisting elliptic islands and chaotic regions, a chaotic trajectory gets captured in the neighborhood of an elliptic island every so often and can stay there for arbitrarily long time; as there the orbit is nearly stable, during such episode $\ln|\sigma_{p,1}(x_0, t)|/t$ can dip arbitrarily close to 0^+. For state space volume non-preserving flows the trajectory can traverse locally contracting regions, and $\ln|\sigma_{p,1}(x_0, t)|/t$ can occasionally go negative; even worse, one never knows whether the asymptotic attractor is periodic or 'chaotic', so any finite time estimate of λ might be dead wrong. exercise 6.3

Résumé

Let us summarize the 'stability' chapters 4 to 6. A neighborhood of a trajectory deforms as it is transported by a flow. In the linear approximation, the stability matrix A describes the shearing / compression / expansion of an infinitesimal neighborhood in an infinitesimal time step. The deformation after a finite time t is described by the Jacobian matrix J^t, whose eigenvalues (stability multipliers) depend on the choice of coordinates.

Floquet multipliers and *eigen-vectors* are intrinsic, invariant properties of finite-time, compact invariant solutions, such as periodic orbits and relative periodic orbits; they are explained in chapter 5. *Stability exponents* [6.1] are the corresponding long-time limits estimated from typical ergodic trajectories.

Finite-time Lyapunov exponents and the associated *principal axes* are defined in (6.8). Oseledec *Lyapunov exponents* are the $t \to \infty$ limit of these.

Commentary

Remark 6.1 Lyapunov exponents are uncool, and ChaosBook does not use them at all. Eigenvectors / eigenvalues are suited to study of iterated forms of a matrix, such as Jacobian matrix J^t or exponential $\exp(tA)$, and are thus a natural tool for study of dynamics. Principal vectors are not, they are suited to study of the matrix J^t itself. The polar (singular value) decomposition is convenient for numerical work (any matrix, square or rectangular, can be brought to such form), as a way of estimating the effective rank of matrix J by separating the large, significant singular values from the small, negligible singular values.

Lorenz [6.2, 6.3, 6.4] pioneered the use of singular vectors in chaotic dynamics. We found the Goldhirsch, Sulem and Orszag [6.1] exposition very clear, and we also enjoyed Hoover and Hoover [6.5] pedagogical introduction to computation of Lyapunov spectra by the method of Lagrange multipliers. Greene and Kim [6.6] discuss singular values vs. Jacobian matrix eigenvalues. While they conclude that "singular values, rather than eigenvalues, are the appropriate quantities to consider when studying chaotic systems," we beg to differ: their Fig. 3, which illustrates various semiaxes of the ellipsoid in the case of Lorenz attractor, as well as the figures in ref. [6.7], are a persuasive argument for *not* using singular values. The covariant vectors are tangent to the attractor, while the principal axes of strain point away from it. It is the perturbations within the attractor that describe the long-time dynamics; these perturbations lie within the subspace spanned by the leading covariant vectors.

That is the first problem with Lyapunov exponents: stretches $\{\sigma_j\}$ are *not related* to the Jacobian matrix J^t eigenvalues $\{\Lambda_j\}$ in any simple way. The eigenvectors $\{u^{(j)}\}$ of strain tensor $J^\top J$ that determine the orientation of the principal axes, are distinct from the Jacobian matrix eigenvectors $\{\mathbf{e}^{(j)}\}$. The strain tensor $J^\top J$ satisfies no multiplicative semigroup property such as (4.20); unlike the Jacobian matrix (5.3), the strain tensor $J^{\top r}J^r$ for the rth repeat of a prime cycle p is not given by a power of $J^\top J$ for the single traversal of the prime cycle p. Under time evolution the covariant vectors map forward as $\mathbf{e}^{(j)} \to J\,\mathbf{e}^{(j)}$ (transport of the velocity vector (4.9) is an example). In contrast, the principal axes have to be recomputed from the scratch for each time t.

If Lyapunov exponents are not dynamical, why are they invoked so frequently? One reason is fear of mathematics: the monumental and therefore rarely read Oseledec [20.6, 20.7] Multiplicative Ergodic Theorem states that the limits (6.7–6.11) exist for almost all points x_0 and vectors $\hat{n}$, and that there are at most d distinct Lyapunov exponents $\lambda_i(x_0)$ as $\hat{n}$ ranges over the tangent space. To intimidate the reader further we note in passing that "moreover there is a fibration of the tangent space $\mathbf{T}_x\mathcal{M}$, $L^1(x) \subset L^2(x) \subset \cdots \subset L^r(x) = \mathbf{T}_x\mathcal{M}$, such that if $\hat{n} \in L^i(x) \setminus L^{i-1}(x)$ the limit (6.7) equals $\lambda_i(x)$." Oseledec proof is important mathematics, but the method is not helpful in elucidating dynamics.

The other reason to study singular vectors is physical and practical: Lorenz [6.2, 6.3, 6.4] was interested in the propagation of errors, i.e., how does a cloud of initial points $x(0) + \delta x(0)$, distributed as a Gaussian with covariance matrix $Q(0) = \langle \delta x(0)\, \delta x(0)^\top \rangle$, evolve in time? For linearized flow with initial isotropic distribution $Q(0) = \epsilon \mathbf{1}$ the answer is given by the left Cauchy-Green strain tensor,

$$Q(t) = \langle \delta x(0)\, J\, J^\top \delta x(0)^\top \rangle = J\, Q(t)\, J^\top = \epsilon\, J\, J^\top \,. \tag{6.12}$$

The deep problem with Lyapunov exponents is that the intuitive definition (6.5) depends on the notion of distance $\| \delta x(t) \|$ between two state space points. The Euclidean (or L^2) distance is natural in the theory of $3D$ continuous media, but what the norm should be for other state spaces is far from clear, especially in high dimensions and for PDEs. As we have shown in sect. 5.3, Floquet multipliers are invariant under all local smooth nonlinear coordinate transformations, they are intrinsic to the flow, and the Floquet eigenvectors are independent of the definition of the norm [6.7]. In contrast, the stretches $\{\sigma_j\}$, and the right/left principal axes depend on the choice of the norm. Appending them to dynamics destroys its invariance.

There is probably no name more liberally and more confusingly used in dynamical systems literature than that of Lyapunov (AKA Liapunov). Singular values / principal axes of strain tensor $J^\top J$ (objects natural to the theory of deformations) and their long-time limits can indeed be traced back to the thesis of Lyapunov [A1.6, 20.6] (English translation [**?**]), and justly deserve sobriquet 'Lyapunov'. Oseledec [20.6] refers to them as 'Liapunov characteristic numbers', and Eckmann and Ruelle [6.11] as 'characteristic exponents'. The natural objects in dynamics are the linearized flow Jacobian matrix J^t, and its eigenvalues and eigenvectors (stability multipliers and covariant vectors). Why should they also be called 'Lyapunov'? The Jacobian matrix eigenvectors $\{\mathbf{e}^{(j)}\}$ (the covariant vectors) are often called 'covariant Lyapunov vectors', 'Lyapunov vectors', or 'stationary Lyapunov basis' [6.12] even though *they are not* the eigenvectors that correspond to the Lyapunov exponents. That's just confusing, for no good reason - the Lyapunov paper [A1.6] is not about the linear stability Jacobian matrix J, it is about $J^\top J$ and the associated principal axes. However, Trevisan [6.7] refers to covariant vectors as 'Lyapunov vectors', and Radons [6.13] calls them 'Lyapunov modes', motivated by thinking of these eigenvectors as a generalization of 'normal modes' of mechanical systems, whereas by ith 'Lyapunov mode' Takeuchi and Chaté [6.14] mean $\{\lambda_j, \mathbf{e}^{(j)}\}$, the set of the ith stability exponent and the associated covariant vector. Kunihiro *et al.* [6.15] call the eigenvalues of stability matrix (4.3), evaluated at a given instant in time, the 'local Lyapunov exponents', and they refer to the set of stability exponents (4.8) for a finite time Jacobian matrix as the 'intermediate Lyapunov exponent', "averaged" over a finite time period. Then there is the unrelated, but correctly attributed 'Lyapunov equation' of control theory, which is the linearization of the 'Lyapunov function', and there is the 'Lyapunov orbit' of celestial mechanics, entirely unrelated to any of objects discussed above.

In short: we do not recommend that you evaluate Lyapunov exponents; compute stability exponents and the associated covariant vectors instead. Cost less and gets you more insight. Whatever you call your exponents, please state clearly how are they being computed. While the Lyapunov exponents are a diagnostic for chaos, we are doubtful of their utility as means of predicting any observables of physical significance. This is the minority position - in the literature one encounters many provocative speculations, especially in the context of foundations of statistical mechanics ('hydrodynamic' modes) and the existence of a Lyapunov spectrum in the thermodynamic limit of spatiotemporal chaotic systems.

Remark 6.2 Matrix decompositions of the Jacobian matrix. A 'polar decomposition' of a matrix or linear operator is a generalization of the factorization of complex number

into the polar form, $z = r\, \exp(\phi)$. Matrix polar decomposition is explained in refs. [6.16, 6.17, 6.18, 6.19]. One can go one step further than the polar decomposition (6.2) into a product of a rotation and a symmetric matrix by diagonalizing the symmetric matrix by a second rotation, and thus express any matrix with real elements in the singular value decomposition (SVD) form

$$J = R_1 D R_2{}^\top \,, \tag{6.13}$$

where D is diagonal and real, and R_1, R_2 are orthogonal matrices, unique up to permutations of rows and columns. The diagonal elements $\{\sigma_1, \sigma_2, \ldots, \sigma_d\}$ of D are the *singular values* of J.

Though singular values decomposition provides geometrical insights into how tangent dynamics acts, many popular algorithms for asymptotic stability analysis (computing Lyapunov spectrum) employ another standard matrix decomposition, the QR scheme [6.20], through which a nonsingular matrix J is (uniquely) written as a product of an orthogonal and an upper triangular matrix $J = QR$. This can be thought as a Gram-Schmidt decomposition of the column vectors of J. The geometric meaning of QR decomposition is that the volume of the d-dimensional parallelepiped spanned by the column vectors of J has a volume coinciding with the product of the diagonal elements of the triangular matrix R, whose role is thus pivotal in algorithms computing Lyapunov spectra [6.21].

Remark 6.3 Numerical evaluation of Lyapunov exponents. There are volumes of literature on numerical computation of the Lyapunov exponents, see for example refs. [6.22, 6.11, 6.23, 6.24]. For early numerical methods to compute Lyapunov vectors, see refs. [6.25, 6.26]. The drawback of the Gram-Schmidt method is that the vectors so constructed are orthogonal by fiat, whereas the stable / unstable eigenvectors of the Jacobian matrix are in general not orthogonal. Hence the Gram-Schmidt vectors are not covariant, i.e., the linearized dynamics does not transport them into the eigenvectors of the Jacobian matrix computed further downstream. For computation of covariant vectors, see refs. [A1.80, 6.28].

6.3 Examples

The reader is urged to study the examples collected here. To return back to the main text, click on [click to return] pointer on the margin.

Example 6.1 ***Lyapunov exponent.*** *Given a 1-dimensional map, consider observable* $\lambda(x) = \ln |f'(x)|$ *and integrated observable*

$$A(x_0, t) = \sum_{k=0}^{n-1} \ln |f'(x_k)| = \ln \left| \prod_{k=0}^{n-1} f'(x_k) \right| = \ln \left| \frac{\partial f^n}{\partial x}(x_0) \right| \,.$$

The Lyapunov exponent is the average rate of the expansion

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'(x_k)| \,.$$

See sect. 6.2 for further details.

Example 6.2 ***Singular values and geometry of deformations:*** *Suppose we are in three dimensions, and the Jacobian matrix J is not singular (yet another confusing usage of word 'singular'), so that the diagonal elements of D in (6.13) satisfy $\sigma_1 \geq \sigma_2 \geq \sigma_3 > 0$. Consider how J maps the unit ball $\mathcal{S} = \{x \in \mathbb{R}^3 \mid x^2 = 1\}$. V is orthogonal (rotation/reflection), so $V^\top \mathcal{S}$ is still the unit sphere: then D maps $\mathcal{S}$ onto ellipsoid $\tilde{\mathcal{S}} = \{y \in \mathbb{R}^3 \mid y_1^2/\sigma_1^2 + y_2^2/\sigma_2^2 + y_3^2/\sigma_3^2 = 1\}$ whose principal axes directions - y coordinates - are determined by V. Finally the ellipsoid is further rotated by the orthogonal matrix U. The local directions of stretching and their images under J are called the right-hand and left-hand singular vectors for J and are given by the columns in V and U respectively: it is easy to check that $Jv_k = \sigma_k u_k$, if v_k, u_k are the k-th columns of V and U.* click to return: p. **??**

Exercises

6.1. **Principal stretches.** Consider $dx = f(x_0 + dx_0) - f(x_0)$, and show that $dx = M dx_0 +$ higher order terms when $\|dx_0\| \ll 1$. (Hint: use Taylor expansion for a vector function.) Here, $\|dx_0\| \equiv \sqrt{dx_0 \cdot dx_0}$ is the norm induced by the usual Euclidean dot (inner) product. Then let $dx_0 = (d\ell) e_i$ and show that $\|dx_0\| = d\ell$ and $\|dx\| = \sigma_i d\ell$. (Christov *et al.* [4.27])

6.2. **Eigenvalues of the Cauchy-Green strain tensor.** Show that $\kappa_i = \sigma_i^2$ using the definition of C, the polar decomposition theorem, and the properties of eigenvalues. (Christov *et al.* [4.27])

6.3. **How unstable is the Hénon attractor?**

(a) Evaluate numerically the Lyapunov exponent λ by iterating some 100,000 times or so the Hénon map

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 - ax^2 + y \\ bx \end{bmatrix}$$

for $a = 1.4$, $b = 0.3$.

(b) Would you describe the result as a 'strange attractor'? Why?

(c) How robust is the Lyapunov exponent for the Hénon attractor? Evaluate numerically the Lyapunov exponent by iterating the Hénon map for $a = 1.39945219$, $b = 0.3$. How much do you now trust your result for part (a) of this exercise?

(d) Re-examine this computation by plotting the iterates, and erasing the plotted points every 1000 iterates or so. Keep at it until the 'strange' attractor vanishes like the smile of the Chesire cat. What replaces it? Do a few numerical experiments to estimate the length of typical transient before the dynamics settles into this long-time attractor.

(e) Use your Newton search routine to confirm existence of this attractor. Compute its Lyapunov exponent, compare with your numerical result from above. What is the itinerary of the attractor.

(f) Would you describe the result as a 'strange attractor'? Do you still have confidence in claims such as the one made for the part (b) of this exercise?

6.4. **Rössler attractor Lyapunov exponents.**

(a) Evaluate numerically the expanding Lyapunov exponent λ_e of the Rössler attractor (2.27).

(b) Plot your own version of figure 6.3. Do not worry if it looks different, as long as you understand why your plot looks the way it does. (Remember the nonuniform contraction/expansion of figure 4.3.)

(c) Give your best estimate of λ_e. The literature gives surprisingly inaccurate estimates - see whether you can do better.

(d) Estimate the contracting Lyapunov exponent λ_c. Even though it is much smaller than λ_e, a glance at the stability matrix (4.31) suggests that you can probably get it by integrating the infinitesimal volume along a long-time trajectory, as in (4.29).

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