Appendix A24

Deterministic diffusion

BASIC NOTIONS of discretizing continuum are introduced: dicretized fields on lattices, lattice derivatives, lattice Laplacians. Invariance of a given theory under (discrete) translations motivates us to consider periodic lattices, and use the eigenmodes of translation generators to diagonalise (discrete Fourier transformations) non-local operators, such as Laplacians, and invert them.

We then use these tools to study in sect. A24.5 some of the simplest examples of deterministic systems that exhibit "deterministic diffusion," the sawtooth and cat maps.

A24.1 Lattice derivatives

In order to set up continuum field-theoretic equations which describe the evolution of spatial variations of fields, we need to define *lattice derivatives*.

Consider a smooth function $\phi(x)$ evaluated on a d-dimensional lattice

$$\phi_\ell = \phi(x)\,, \qquad x = a\ell = \text{lattice point}\,, \quad \ell \in \mathbf{Z}^d\,, \tag{A24.1}$$

where a is the lattice spacing. Each set of values of $\phi(x)$ (a vector ϕ_ℓ) is a possible lattice state (or 'configuration'). Assume the lattice is hyper-cubic, and let $\hat{n}_\mu \in \{\hat{n}_1, \hat{n}_2, \cdots, \hat{n}_d\}$ be the unit lattice cell vectors pointing along the d positive directions. The forward *lattice derivative* is then

$$(\partial_\mu \phi)_\ell = \frac{\phi(x + a\hat{n}_\mu) - \phi(x)}{a} = \frac{\phi_{\ell+\hat{n}_\mu} - \phi_\ell}{a}\,. \tag{A24.2}$$

The backward *lattice derivative* is defined as the transpose

$$(\partial_\mu \phi)^\top_\ell = \frac{\phi(x - a\hat{n}_\mu) - \phi(x)}{a} = \frac{\phi_{\ell-\hat{n}_\mu} - \phi_\ell}{a}\,. \tag{A24.3}$$

Anything else with the correct $a \to 0$ limit would do, but this is the simplest choice. We can rewrite the lattice derivative as a linear operator, by introducing the *stepping operator* in the direction μ

$$\left(\sigma_\mu\right)_{\ell j} = \delta_{\ell+\hat{n}_\mu, j}\,. \tag{A24.4}$$

As σ will play a central role in what follows, it pays to understand what it does.

In computer dicretizations, the lattice will be a finite d-dimensional hyper-cubic lattice

$$\phi_\ell = \phi(x)\,, \qquad x = a\ell = \text{lattice point}\,, \quad \ell \in (\mathbf{Z}/N)^d\,, \tag{A24.5}$$

where a is the lattice spacing and there are N^d points in all. For a hyper-cubic lattice the translations in different directions commute, $\sigma_\mu\sigma_\nu = \sigma_\nu\sigma_\mu$, so it is sufficient to understand the action of (A24.4) on a 1-dimensional lattice.

Let us write down σ for the 1-dimensional case in its full $[N\times N]$ matrix glory. Writing the finite lattice stepping operator (A24.4) as an 'upper shift' matrix,

$$\sigma = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & & \ddots & \\ & & & & 0 & 1 \\ 0 & & & & & 0 \end{bmatrix}, \tag{A24.6}$$

is no good, as σ so defined is nilpotent, and after N steps the particle marches off the lattice edge, and nothing is left, $\sigma^N = 0$. A sensible way to approximate an infinite lattice by a finite one is to replace it by a lattice periodic in each $\hat{n}_\mu$ direction. On a *periodic lattice* every point is equally far from the 'boundary' $N/2$ steps away, the 'surface' effects are equally negligible for all points, and the stepping operator acts as a cyclic permutation matrix

$$\sigma = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & & \ddots & \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{bmatrix}, \tag{A24.7}$$

with '1' in the lower left corner assuring periodicity.

Applied to the lattice state $\phi = (\phi_1, \phi_2, \cdots, \phi_N)$, the stepping operator translates the state by one site, $\sigma\phi = (\phi_2, \phi_3, \cdots, \phi_N, \phi_1)$. Its transpose translates the configuration the other way, so the transpose is also the inverse, $\sigma^{-1} = \sigma^T$. The partial lattice derivative (A24.3) can now be written as a multiplication by a matrix:

$$\partial_\mu\phi_\ell = \frac{1}{a}\left(\sigma_\mu - \mathbf{1}\right)_{\ell j}\phi_j\,.$$

In the 1-dimensional case the $[N{\times}N]$ matrix representation of the lattice derivative is:

$$\partial = \frac{1}{a}\begin{bmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ 1 & & & & & -1 \end{bmatrix}. \tag{A24.8}$$

To belabor the obvious: On a finite lattice of N points a derivative is simply a finite $[N \times N]$ matrix. Continuum field theory is a world in which the lattice is so fine that it looks smooth to us. Whenever someone calls something an "operator," think "matrix." For finite-dimensional spaces a linear operator *is* a matrix; things get subtler for infinite-dimensional spaces.

A24.1.1 Lattice Laplacian

In the continuum, integration by parts moves ∂ around, $\int [dx]\phi^\top \partial^2 \phi \to -\int [dx]\partial\phi^\top \cdot \partial\phi$; on a lattice this amounts to a matrix transposition

$$\left[\left(\sigma_\mu - \mathbf{1}\right)\phi\right]^\top \cdot \left[\left(\sigma_\mu - \mathbf{1}\right)\phi\right] = \phi^\top \cdot \left(\sigma_\mu^{-1} - \mathbf{1}\right)\left(\sigma_\mu - \mathbf{1}\right)\phi\,.$$

If you are wondering where the "integration by parts" minus sign is, it is there in discrete case at well. It comes from the identity

$$\partial^\top = \frac{1}{a}\left(\sigma^{-1} - \mathbf{1}\right) = -\sigma^{-1}\frac{1}{a}\left(\sigma - \mathbf{1}\right) = -\sigma^{-1}\partial\,.$$

The symmetric (self-adjoint) combination $\Box = -\partial^\top\partial$

$$\begin{aligned} \Box &= -\frac{1}{a^2}\sum_{\mu=1}^{d}\left(\sigma_\mu^{-1} - \mathbf{1}\right)\left(\sigma_\mu - \mathbf{1}\right) = \frac{1}{a^2}\sum_{\mu=1}^{d}\left(\sigma_\mu^{-1} + \sigma_\mu - 2\,\mathbf{1}\right) \\ &= \frac{1}{a^2}\left(T - 2d\mathbf{1}\right) \end{aligned} \tag{A24.9}$$

is the *lattice Laplacian*. We shall show below that this Laplacian has the correct continuum limit. In the 1-dimensional case the $[N \times N]$ matrix representation of the lattice Laplacian is:

$$\Box = \frac{1}{a^2}\begin{bmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & 1 & & \ddots & \\ & & & & & 1 \\ 1 & & & & 1 & -2 \end{bmatrix}. \tag{A24.10}$$

The lattice Laplacian measures the second variation of a field ϕ_ℓ across three neighboring sites: it is spatially *non-local*. You can easily check that it does what the second derivative is supposed to do by applying it to a parabola restricted to the lattice, $\phi_\ell = \phi(a\ell)$, where $\phi(a\ell)$ is defined by the value of the continuum function $\phi(x) = x^2$ at the lattice point $x_\ell = a\ell$.

The Euclidean free scalar particle propagator can thus be written as

$$\Delta = \frac{1}{\mathbf{1} - \frac{h}{s}a^2\Box}\,. \tag{A24.11}$$

In what follows it will be convenient to reinterpret and rescale this drunken-walk propagator Δ, and consider instead the "free field action" of form

$$S[\phi] = -\frac{1}{2}\phi^\top \cdot M^{-1} \cdot \phi\,. \tag{A24.12}$$

where the "free" or "bare" massive scalar propagator M is parametrized as

$$M = \frac{1}{m^2 - \Box}\,. \tag{A24.13}$$

What this parametrization says is that the mass squared m^2 of the Euclidean scalar particle is proportional to $m^2 \sim s/h$: the heavier the particle, the less likely it is to hop, the more likely is it to stop.

A24.1.2 Inverting the Laplacian

Evaluation of perturbative corrections requires that we come to grips with the "free" or "bare" propagator M. While the Laplacian is a simple difference operator (A24.10), the propagator is a messier object. A way to compute is to start expanding the propagator M as a power series in the Laplacian

$$M = \frac{1}{m^2 - \Box} = \frac{1}{m^2}\sum_{k=0}^{\infty}\frac{1}{m^{2k}}\Box^k\,. \tag{A24.14}$$

As $\Box$ is a finite matrix, the expansion is convergent for sufficiently large m^2. To get a feeling for what is involved in evaluating such series, evaluate $\Box^2$ in the 1-dimensional case:

$$\Box^2 = \frac{1}{a^4}\begin{bmatrix} 6 & -4 & 1 & & & 1 & -4 \\ -4 & 6 & -4 & 1 & & & 1 \\ 1 & -4 & 6 & -4 & 1 & & \\ & 1 & -4 & \ddots & & & 1 \\ 1 & & & & & 6 & -4 \\ -4 & 1 & & & 1 & -4 & 6 \end{bmatrix}\,. \tag{A24.15}$$

What $\Box^3$, $\Box^4$, $\cdots$ contributions look like is now clear; as we include higher and higher powers of the Laplacian, the propagator matrix fills up; while the *inverse* propagator is differential operator connecting only the nearest neighbors, the propagator is integral, *non-local* operator, connecting every lattice site to any other lattice site. Due to the periodicity, these are all Toeplitz matrices, meaning that each successive row is a one-step cyclic shift of the preceding one. In statistical mechanics, M is the (bare) 2-point correlation. In quantum field theory, it is called a propagator.

These matrices can be evaluated as is, on the lattice, and sometime it is evaluated this way, but in case at hand a wonderful simplification follows from the observation that the lattice action is translationally invariant. We show how this works in sect. A24.2.

A24.2 Periodic lattices

Our task now is to transform M into a form suitable to explicit evaluation.

Consider the effect of a lattice translation $\phi \to \sigma\phi$ on the matrix polynomial

$$S[\sigma\phi] = -\frac{1}{2}\phi^\top \left(\sigma^\top M^{-1}\sigma\right)\phi\,.$$

As M^{-1} is constructed from σ and its inverse, M^{-1} and σ commute, and $S[\phi]$ is invariant under translations,

$$S[\sigma\phi] = S[\phi] = -\frac{1}{2}\phi^\top \cdot \frac{1}{M} \cdot \phi\,. \tag{A24.16}$$

If a function defined on a vector space commutes with a linear operator σ, then the eigenvalues of σ can be used to decompose the ϕ vector space into invariant subspaces. For a hyper-cubic lattice the translations in different directions commute, $\sigma_\mu\sigma_\nu = \sigma_\nu\sigma_\mu$, so it is sufficient to understand the spectrum of the 1-dimensional stepping operator (A24.7).

To develop a feeling for how this reduction to invariant subspaces works in practice, let us proceed cautiously, by expanding the scope of our deliberations to a lattice consisting of 2 points.

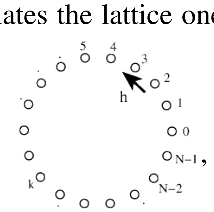
example A24.1
p. 1197

A24.3 Discrete Fourier transforms

Let us generalize this reduction to a 1-dimensional periodic lattice with N sites.

Each application of σ translates the lattice one step; in N steps the lattice is back in the original state

$$\sigma^N = \mathbf{1} \tag{A24.17}$$



so the eigenvalues of σ are the N distinct Nth roots of unity

$$\sigma^N - \mathbf{1} = \prod_{k=0}^{N-1}(\sigma - \omega^k \mathbf{1}) = 0\,, \qquad \omega = e^{i\frac{2\pi}{N}}\,. \tag{A24.18}$$

As the eigenvalues are all distinct and N in number, the space is decomposed into N 1-dimensional subspaces. The general theory (expounded in appendix A10.2) associates with the kth eigenvalue of σ a projection operator that projects a state ϕ onto kth eigenvector of σ,

$$P_k = \prod_{j\neq k} \frac{\sigma - \omega^j \mathbf{1}}{\omega^k - \omega^j}\,. \tag{A24.19}$$

A factor $(\sigma - \omega^j \mathbf{1})$ kills the jth eigenvector φ_j component of an arbitrary vector in expansion $\phi = \cdots + \tilde{\phi}_j \varphi_j + \cdots$. The above product kills everything but the eigen-direction φ_k, and the factor $\prod_{j\neq k}(\omega^k - \omega^j)$ ensures that P_k is normalized as a projection operator. The set of the projection operators is complete,

$$\sum_k P_k = \mathbf{1}\,, \tag{A24.20}$$

and orthonormal

$$P_k P_j = \delta_{kj} P_k \qquad \text{(no sum on } k)\,. \tag{A24.21}$$

In the case of discrete translational invariance, or cyclic group C_N, it is customary to write out the projection operator (A24.19) as a character-weighted sum, see example A24.2.

As any matrix function $M = M(\sigma)$ of the translation generator σ takes a scalar value on the kth subspace,

$$M(\sigma)\, P_k = M(\omega^k)\, P_k\,, \tag{A24.22}$$

the projection operators *diagonalize* the matrix M, $P_j\, M(\sigma)\, P_k = M(\omega^k)\, P_k\, \delta_{jk}$.

example A24.2
p. 1197

A24.3.1 Eigenvectors of the translation operator

While constructing explicit eigenvectors is usually not a the best way to fritter one's youth away, as choice of basis is largely arbitrary, and all of the content of the theory is in the projection operators (see appendix A10.2), in case at hand the eigenvectors are so simple that we can construct and verify the solutions of the eigenvalue condition

$$\sigma\, \varphi_k = \omega^k \varphi_k \tag{A24.23}$$

by hand:

$$\frac{1}{\sqrt{N}}\begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & & \ddots & \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{bmatrix}\begin{bmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \omega^{3k} \\ \vdots \\ \omega^{(N-1)k} \end{bmatrix} = \omega^k \frac{1}{\sqrt{N}}\begin{bmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \omega^{3k} \\ \vdots \\ \omega^{(N-1)k} \end{bmatrix}$$

In words: the cyclic translation generator σ shifts all components by one, and the original vector is recovered by factoring out the common factor ω^k. The $1/\sqrt{N}$ factor normalizes φ_k to a complex unit vector,

$$\begin{aligned} \varphi_k^\dagger \cdot \varphi_k &= \frac{1}{N}\sum_{k=0}^{N-1} 1 \;=\; 1\,, \qquad \text{(no sum on } k\text{)} \\ \varphi_k^\dagger &= \frac{1}{\sqrt{N}}\left(1, \omega^{-k}, \omega^{-2k}, \cdots, \omega^{-(N-1)k}\right). \end{aligned} \tag{A24.24}$$

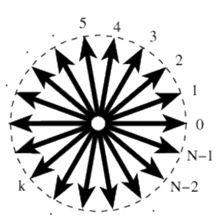
The eigenvectors are orthonormal

$$\varphi_k^\dagger \cdot \varphi_j = \delta_{kj}\,, \tag{A24.25}$$

as the explicit evaluation of $\varphi_k^\dagger \cdot \varphi_j$ yields the *Kronecker (circular) delta function for a periodic lattice*

$$\delta_{kj} = \frac{1}{N}\sum_{\ell=0}^{N-1} e^{i\frac{2\pi}{N}(k-j)\ell}\,. \tag{A24.26}$$

The sum is over the N unit vectors pointing at a uniform distribution of points on the complex unit circle,



,

they cancel each other unless $k = j \pmod N$, in which case each term in the sum equals 1.

By the eigenvector condition (A24.23), any matrix function $M = M(\sigma)$ of the translation generator σ takes a scalar value on the kth subspace,

$$M(\sigma)\,\varphi_k = M(\omega^k)\,\varphi_k\,, \tag{A24.27}$$

i.e., in the eigenvector basis, M is a diagonal matrix.

The $[N\times N]$ projection operator matrix elements can be expressed in terms of the eigenvectors (A24.23), (A24.24) as

$$(P_k)_{\ell\ell'} = (\varphi_k)_\ell(\varphi_k^\dagger)_{\ell'} = \frac{1}{N}e^{i\frac{2\pi}{N}(\ell-\ell')k}\,, \qquad \text{(no sum on } k\text{)}\,. \tag{A24.28}$$

The completeness (A24.20) follows from (A24.26), and the orthonormality (A24.21) from (A24.25).

$\tilde{\phi}_k$, the projection of the N-dimensional state (i.e., vector) ϕ on the kth subspace is given by

$$\begin{aligned}
(P_k\cdot\phi)_\ell &= \tilde{\phi}_k\,(\varphi_k)_\ell\,, \qquad \text{(no sum on } k\text{)} \\
\tilde{\phi}_k &= \varphi_k^\dagger\cdot\phi = \frac{1}{\sqrt{N}}\sum_{\ell=0}^{N-1} e^{-i\frac{2\pi}{N}k\ell}\phi_\ell
\end{aligned} \tag{A24.29}$$

The N-dimensional vector $\tilde{\phi}$ of "wavenumbers" (discretized spatial coordinates), or "frequencies," "eigen-energies" (discretized time evolution steps) $\tilde{\phi}_k$ is the *discrete Fourier transform* of state (vector) ϕ. Hopefully rediscovering it this way helps you a little toward understanding why Fourier transforms are full of $e^{ix\cdot p}$ factors (they are eigenvalues of generators of translations; σ for a discrete lattice, $\partial/\partial x$ for continuum), and that they are the natural set of basis functions when a theory is translationally invariant.

example A24.2
p. 1197

A24.3.2 Discrete Fourier transform operator

The $[N\times N]$ matrix $\mathcal{F}_{jk} = N^{-\frac{1}{2}}\omega^{jk}$, $j,k = 0,1,2,\cdots,N-1$, formed from column eigenvectors (A24.23),

$$\mathcal{F} = \frac{1}{\sqrt{N}}\begin{bmatrix}
1 & 1 & 1 & \dots & 1 & 1 \\
1 & \omega & \omega^2 & \dots & \omega^{N-2} & \omega^{N-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \omega^k & \omega^{2k} & \dots & \omega^{(N-2)k} & \omega^{(N-1)k} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \omega^{N-2} & \omega^{2(N-2)} & \dots & \omega^{(N-2)(N-2)} & \omega^{(N-1)(N-2)} \\
1 & \omega^{N-1} & \omega^{2(N-1)} & \dots & \omega^{(N-2)(N-1)} & \omega^{(N-1)(N-1)}
\end{bmatrix}, \tag{A24.30}$$

is the *discrete Fourier transform operator* (remember, in the discretized world 'operator' is a synonym for 'matrix'). From the orthogonality of eigenvectors (A24.25) it follows that $\mathcal{F}$ is a unitary matrix, with $\det\mathcal{F} = 1$, and

$$\mathcal{F}\mathcal{F}^\dagger = \mathbf{1}\,. \tag{A24.31}$$

The operator $\mathcal{F}^\dagger$ is thus the *inverse Fourier transform.* The discrete Fourier transform (A24.29) of a state (vector) ϕ is given by

$$\tilde{\phi} = \mathcal{F}^\dagger\phi\,, \tag{A24.32}$$

i.e., Fourier transformation rearranges components of vector ϕ into averages over all components (A24.29), weighted by complex phases $\exp(i2\pi\ell/N)$ in all possible ways.

example A24.3
p. 1198

The complex function $\tilde{\phi}$ is can sometimes be interpreted as an 'amplitude function', with the square of its magnitude $(\tilde{\phi}^\dagger \cdot \tilde{\phi})$ then interpreted as the corresponding 'total probability'

$$\phi^\dagger \cdot \phi = \tilde{\phi}^\dagger \cdot \tilde{\phi} \,. \tag{A24.33}$$

The fact that this is the same if evaluated with ϕ or with its Fourier transform $\tilde{\phi}$ is known as the "Parseval's identity."

Furthermore, by (A24.27), discrete Fourier transform diagonalizes every translationally invariant matrix function M, i.e., any matrix that commutes with the translation operator, $[\sigma, M] = 0$. To show that, sandwich M with the identity $\mathbf{1} = \mathcal{F}\mathcal{F}^\dagger$:

$$M = \mathbf{1}M\mathbf{1} = \mathcal{F}\left(\mathcal{F}^\dagger M \mathcal{F}\right)\mathcal{F}^\dagger = \mathcal{F}\tilde{M}\mathcal{F}^\dagger \,.$$

The matrix

$$\tilde{M} = \mathcal{F}^\dagger M \mathcal{F} \tag{A24.34}$$

is the Fourier transform of M. The form of any translation-invariant function, such as (A24.33), or the invariant function (A24.16) does not change under $\phi \to \tilde{\phi}$ transformation, and it does not matter whether we compute in the Fourier space, or in the configuration space that we started out with. For example, the trace of M is the same in either representation

$$\mathrm{tr}\, M = \mathrm{tr}\, \mathcal{F}\tilde{M}\mathcal{F}^\dagger = \mathrm{tr}\, \tilde{M}\mathcal{F}^\dagger\mathcal{F} = \mathrm{tr}\, \tilde{M} \,,$$

but, if M commutes with the translation operator σ, the Fourier transform $\mathrm{tr}\, \tilde{M}$ is diagonal and trivial to compute. By same reasoning it follows that $\mathrm{tr}\, M^n = \mathrm{tr}\, \tilde{M}^n$, and from the $\mathrm{tr}\ln = \ln \mathrm{tr}$ relation that $\det M = \det \tilde{M}$. In fact, any scalar combination of ϕ's, J's and couplings, such as the partition function $Z[J]$, has exactly the same form in the configuration and the Fourier space.

Suppose you have two translationally invariant matrices A, B. Evaluating their product AB is a matrix computation. However, evaluating the product in the Fourier space is a simple scalar multiplication of their diagonal elements:

$$(\tilde{AB})_{kk'} = (\mathcal{F}^\dagger A\, B \mathcal{F})_{kk'} = \tilde{A}_k \tilde{B}_k \delta_{kk'} \tag{A24.35}$$

The continuum Fourier transform version of this relation is called the "convolution theorem."

OK. But what's the payback?

A24.3.3 Lattice Laplacian diagonalized

We can now use the Fourier transform (A24.34) to convert matrix functions of the σ matrix into scalars. If M *commutes* with σ, then $(\tilde{M})_{kk'} = \tilde{M}_k \delta_{kk'}$ is a diagonal matrix, where the matrix M acts as a multiplication by the scalar $\tilde{M}_k$ on the kth subspace. For example, for the 1-dimensional version of the lattice Laplacian matrix (A24.9), the eigenvalue condition (A24.23) yields the diagonalized Laplacian in the Fourier space,

$$\begin{aligned} \tilde{\Box}_{kk'} = (\mathcal{F}^\dagger \Box \mathcal{F})_{kk'} &= \frac{2}{a^2}\left(\frac{1}{2}(\omega^{-k} + \omega^k) - 1\right)\delta_{kk'} \\ &= \frac{2}{a^2}\left(\cos\left(\frac{2\pi}{N}k\right) - 1\right)\delta_{kk'} \,. \end{aligned} \tag{A24.36}$$

In the kth subspace the bare propagator is simply a number, and, in contrast to the mess generated by the configuration space inversion (A24.14), there is nothing to inverting M to M^{-1}:

$$(\varphi^\dagger_{\mathbf{k}} \cdot M^{-1} \cdot \varphi_{\mathbf{k}'}) = \frac{\delta_{\mathbf{k}\mathbf{k}'}}{m^2 - 2\sum_{\mu=1}^{d}\left(\cos\left(\frac{2\pi}{N}k_\mu\right) - 1\right)} \,, \tag{A24.37}$$

where $\mathbf{k} = (k_1, k_2, \cdots, k_d)$ is a d-dimensional vector in the N^d-dimensional dual lattice, i.e., the discretized "momentum" or "frequency" space.

Going back to the partition function and sticking in the factors of $\mathbf{1}$ into the bilinear part of the interaction, we replace the spatial source field J by its Fourier transform $\tilde{J}$, and the spatial propagator M by the diagonalized Fourier transformed $\tilde{G}_0$

$$J^\dagger \cdot M \cdot J = J^\dagger \cdot \mathcal{F}\left(\mathcal{F}^\dagger M \mathcal{F}\right)\mathcal{F}^\dagger \cdot J = \tilde{J}^\dagger \cdot \tilde{G}_0 \cdot \tilde{J} \,. \tag{A24.38}$$

A24.4 Continuum field theory

The lattice Laplacian kth Fourier component (A24.36) is

$$\begin{aligned} \tilde{\Box}_{kk} &= \frac{2}{a^2}\left(\cos\left(\frac{2\pi}{N}k\right) - 1\right) \\ &= -\left(\frac{2\pi}{aN}\right)^2 k^2 + \frac{1}{12}\left(\frac{2\pi}{aN}\right)^4 a^2 k^4 - O(k^6) \,. \end{aligned} \tag{A24.39}$$

The quartic term can be neglected for low wave numbers $k \ll N$, i.e., low momenta, $p_\mu = 2\pi k_\mu / L$, where $aN = L$ is the lattice size.

In the continuum limit the probability to land in the kth cell is replaced by a probability density, $\phi_k = a^d \phi(x_k) \to (dx)^d \phi(x)$. After rescaling the wave-number k into momentum p, we obtain the continuum version of the scalar propagator

$$\Delta(x, y) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip\cdot(x-y)}}{m^2 + p^2} \,. \tag{A24.40}$$

A24.5 Diffusion in sawtooth and cat maps

(R. Artuso)

In this section we will deal with the prototype example of chaotic Hamiltonian maps, hyperbolic toral automorphisms. Diffusive properties will arise in considering such maps acting on the cylinder or over $\mathbb{R}^2$, while the dynamics restricted to the fundamental domain involves maps on $\mathbf{T}^2$ (two–dimensional torus). An Anosov map thus corresponds to the action of a matrix in $SL_2(\mathbb{N})$ with unit determinant and absolute value of the trace bigger than 2.

Maps of this kind are as examples of genuine Hamiltonian chaotic evolution. They admit simple finite Markov partitions, which paves the way to a good symbolic dynamics. Within the framework of Hamiltonian dynamical systems the role of hyperbolic linear automorphisms is analogous to piecewise linear Markov maps: their symbolic dynamics can be encoded in a grammatically simple way, and their linearity leads to uniformity of cycle stabilities.

chapter 28

We will consider the "two-coordinates" representation for them

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix}$$

with

$$M = \begin{bmatrix} 0 & 1 \\ -1 & K+2 \end{bmatrix}$$

which allows considering their extension on a cylinder phase space ($[-1/2, 1/2) \times \mathbb{R}$) in a natural way. So it is natural to study diffusion properties along the y direction.

Though Markov partitions encode the symbolic dynamics in the simplest possible way, they are not well suited to deal with diffusion, as the jumping factor is not related in a simple way to the induced symbol sequence. To this end the following linear code is quite natural: before describing it let us fix the notations: χ will denote the trace of the map ($\chi = K + 2$): the leading eigenvalue will be denoted by $\lambda = (\chi + \sqrt{D})/2$, where $D = \chi^2 - 4$. In principle the code (and the problem of diffusion) can be also considered for real values of K (thus loosing continuity of the torus map when K in not an integer): we will remark in what follows that results which are exact for $K \in \mathbb{N}$ are only approximate for generic K.

The cardinality of the alphabet is determined by the parameter K: the letters are integer numbers, whose absolute values does not exceed $Int(1 + \chi/2)$ (see figure A24.1 for the case $K = 2$). The code is linear, as, given a bi-infinite sequence $\{x_i\}_{i\in\mathbb{N}}$

$$b_t \stackrel{\text{def}}{=} \left[(K+2)x_t - x_{t-1} + \frac{1}{2}\right] \quad , \tag{A24.41}$$

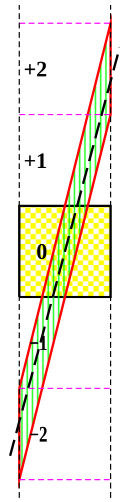


Figure A24.1: The elementary cell for the torus map $[-1/2, 1/2]$ (checkered yellow) together with its image, in green ($K = 2$): symbols refer to the linear code. The dashed line through the origin gives the direction of the unstable manifold. Though hardly understandable from the scale of the picture the unstable manifold is not parallel to the image sides.

$[\dots]$ denoting the integer part, while the inversion formula (once a bi-infinite symbolic string $\{b_i\}_{i\in\mathbb{N}}$ is given), reads

$$x_t = \frac{1}{\sqrt{D}} \sum_{s\in\mathbb{N}} \lambda^{-|t-s|} b_s \quad , \tag{A24.42}$$

As the x coordinate lives in the interval $[-1/2, 1/2)$, (A24.42) induces a condition of allowed symbol sequences: $\{b_i\}_{i\in\mathbb{N}}$ will be an admissible orbit if

$$\frac{1}{2} \leq \frac{1}{\sqrt{D}} \sum_{s\in\mathbb{N}} \lambda^{-|t-s|} b_s < \frac{1}{2} \quad . \tag{A24.43}$$

By (A24.41) and (A24.42) it is easy to observe that periodic orbits and allowed periodic symbol sequences are in one-to-one correspondence. From (A24.43) we get the condition that a $\{b_i\}_{i=1,\dots,T}$ sequence corresponds to a T–periodic orbit of the map

$$|A_n b_t + A_{n-1}(b_{t+1} + b_{t-1}) + \cdots + A_0(b_{t+n} + b_{t-n})| < \frac{B_n}{2} \qquad \forall t = 1, \dots, T$$

when $T = 2n + 1$, and

$$|C_n b_t + C_{n-1}(b_{t+1} + b_{t-1}) + \cdots + C_0(b_{t+n})| < \frac{D_n}{2} \qquad \forall t = 1, \dots, T \tag{A24.44}$$

when $T = 2n$ where

exercise A24.3
exercise A24.4

$$\begin{aligned} B_k &= \lambda^k(\lambda - 1) + \lambda^{-k}(\lambda^{-1} - 1) & \qquad A_k &= \frac{\lambda^{k+1} + \lambda^{-k}}{\lambda + 1} \\ D_k &= (\lambda^k - \lambda^{-k})(\lambda - \lambda^{-1}) & C_k &= \lambda^k + \lambda^{-k} \end{aligned} \tag{A24.45}$$

The pruning rules (A24.44) admit a simple geometric interpretation: a lattice point $\boldsymbol{b} \in \mathbb{N}^T$ identifies a T–periodic point of the map if $\boldsymbol{b} \in \mathcal{P}_T$ where

$$\mathcal{P}_T \stackrel{\text{def}}{=} \{\boldsymbol{x} \in \mathbb{R}^T : \quad \left\{ \begin{array}{c} |a_1 x_1 + \cdots + a_T x_T| < e_T \\ \vdots \\ |a_2 x_1 + \cdots + a_1 x_T| < e_T \end{array} \right\} \tag{A24.46}$$

and

$$\begin{array}{ll} a_1 \ldots a_T = A_0 A_1 \ldots A_{n-1} A_n A_{n-1} \ldots A_0 & e_T = B_n/2 \\ a_1 \ldots a_T = C_1 \ldots C_{n-1} C_n C_{n-1} \ldots C_1 C_0 & e_T = D_n/2 \end{array} \tag{A24.47}$$

for $T = 2n{+}1$ or $T = 2n$, respectively, Thus $\mathcal{P}_T$ is a measure polytope [7], obtained by deforming a T–cube. This is the key issue of this appendix: though the map is endorsed with a most remarkable symbolic dynamics, the same is hardly fit to deal with transport properties, as the rectangles that define the partition are not directly connected to translations once the map is unfolded to the cylinder. The partition connected to the linear code (see figure A24.1) on the other side is most natural when dealing with transport, though its not being directly related to invariant manifolds leads to a multitude of pruning rules (which in the present example bear a remarkable geometric interpretation, which is not to be expected as a generic feature).

exercise A24.5

We will denote by $\mathcal{N}_{n,s}$ the number of periodic points of period n with jumping number s. A way to compute D for cat maps is provided by

$$D = \lim_{n\to\infty} D_n \qquad D_n = \frac{1}{n\mathcal{N}_n} \sum_{k=1}^{p(n)} k^2 \mathcal{N}_{n,k} \tag{A24.48}$$

where $\mathcal{N}_n$ is the number of periodic points of period n, $p(n)$ is the highest jumping number of n–periodic orbits and we employed

$$\left|\det\left(\mathbf{1} - \boldsymbol{J}_x^{(n)}\right)\right| = (\lambda^n - 1)(1 - \lambda^{-n}) = \mathcal{N}_n$$

which is valid for cat maps.

Sums can be converted into integrals by using Poisson summation formula: we define

$$f_T(n) = \left\{ \begin{array}{ll} (n_1 + \cdots + n_T)^2 & n \in \mathcal{P}_T \bigcap \mathbb{N}^T \\ 0 & otherwise \end{array} \right.$$

and

$$\tilde{f}_T(\xi) = \int_{\mathbb{R}^T} dx\, e^{i(x,\xi)} f_T(x)$$

From Poisson summation formula we have that

$$D_T = \frac{1}{T\mathcal{N}_T} \sum_{n\in\mathbb{N}^T} \tilde{f}_T(2\pi n) \tag{A24.49}$$

The quasilinear estimate for D_T amounts to considering the $n = 0$ contribution to (A24.49):

$$D_T^{(q.l.)} = \int_{\mathcal{P}_T} dx\, (x_1 + x_2 + \cdots + x_T)^2 \tag{A24.50}$$

The evaluation of (A24.50) requires introducing a coordinate transformation in symbolic space in which $\mathcal{P}_T$ is transformed in a T–cube. This is equivalent to finding the inverse of the matrix A:

$$A \stackrel{\text{def}}{=} \begin{pmatrix} a_1 & a_2 & \cdots & a_{T-1} & a_T \\ a_T & a_1 & \cdots & a_{T-2} & a_{T-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_3 & a_4 & \cdots & a_1 & a_2 \\ a_2 & a_3 & \cdots & a_T & a_1 \end{pmatrix} \quad . \tag{A24.51}$$

First of all let us observe that A is a circulant matrix, so that its determinant is the product of T factors, each of the form $f(\epsilon_j) = a_1 + \epsilon_j a_2 + \cdots + a_T \epsilon_j^{T-1}$, where ϵ_j is a Tth root of unity. By using (A24.45) it is possible to see that

$$f(\epsilon_j) = \begin{cases} \frac{\epsilon_j^{n+1} B_n}{(\lambda\epsilon_j - 1)(1-\lambda^{-1}\epsilon_j)} & T = 2n+1 \\ \frac{\epsilon_j^{n} D_n}{(\lambda\epsilon_j - 1)(1-\lambda^{-1}\epsilon_j)} & T = 2n \end{cases}$$

so that

$$|\det A| = \frac{(2e_T)^T}{\lambda^T + \lambda^{-T} - 2} \tag{A24.52}$$

By using the results coming from the former exercise we can finally express A^{-1} via exercise A24.6

$$\tilde{C} A^{-1} = \frac{1}{B_n^T} \begin{pmatrix} \chi & -1 & \cdots & 0 & -1 \\ -1 & \chi & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \chi & -1 \\ -1 & 0 & \cdots & -1 & \chi \end{pmatrix} \tag{A24.53}$$

where

$$\tilde{C} = \begin{pmatrix} \mathbf{0} & \mathbb{1}_{n+1} \\ \mathbb{1}_n & \mathbf{0} \end{pmatrix} \quad .$$

if $T = 2n+1$ and

$$\tilde{K} A^{-1} = \frac{1}{D_n^T} \begin{pmatrix} \chi & -1 & \cdots & 0 & -1 \\ -1 & \chi & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \chi & -1 \\ -1 & 0 & \cdots & -1 & \chi \end{pmatrix} \tag{A24.54}$$

where

$$\tilde{K} = \begin{pmatrix} \mathbf{0} & \mathbb{1}_{n} \\ \mathbb{1}_n & \mathbf{0} \end{pmatrix} \quad .$$

if $T = 2n$. As a first check of quasilinear estimates let's compute the volume of $\mathcal{P}_T$:

$$\begin{aligned} Vol(\mathcal{P}_T) &= \int_{\mathcal{P}_T} dx_1\, dx_2 \ldots dx_T = \frac{1}{|\det A|} \int_{-e_T}^{e_T} \cdots \int_{e_T}^{e_T} d\xi_1 \ldots d\xi_T \\ &= \lambda^T + \lambda^{-T} - 2 \end{aligned} \tag{A24.55}$$

In an analogous way we may compute the quasilinear estimate for $\mathcal{N}_{T,k}$ exercise A24.7

$$
\begin{aligned}
\mathcal{N}^{(q.l.)}_{T,k} &= \int_{\mathcal{P}_T} dx_1 \ldots dx_T\, \delta(x_1 + \ldots x_T - k) \\
&= \frac{\lambda^T + \lambda^{-T} - 2}{(2e_T)^T} \int_{-\infty}^{\infty} d\alpha\, e^{-2\pi i\alpha k} \int_{-e_T}^{e_T} \cdots \int_{-e_T}^{e_T} d\xi_1 \ldots d\xi_T\, e^{\frac{2\pi i\alpha\chi}{2e_T}(\xi_1+\cdots+\xi_T)} \\
&= \frac{2}{\pi\chi}(\lambda^T + \lambda^{-T} - 2) \int_0^{\infty} dy\, \cos\left(\frac{2qy}{\chi}\right)\left(\frac{\sin y}{y}\right)^T \qquad \text{(A24.56)}
\end{aligned}
$$

where we have used $x_1+\cdots+x_T = (\chi/(2e_T))(\xi_1+\cdots+\xi_T)$ (cfr. (A24.53),(A24.54)).

We are now ready to evaluate the quasilinear estimate fo the diffusion coefficient

$$
D^{(q.l.)}_T = \frac{1}{\pi\chi T} \int_{-T\chi/2}^{T\chi/2} dz\, z^2 \int_0^{\infty} dy\, \cos\left(\frac{2zy}{\chi}\right)\left(\frac{\sin y}{y}\right)^T \qquad \text{(A24.57)}
$$

(where the bounds on the jumping number again come easily from (A24.53),(A24.54)). By dropping terms vanishing as $T \mapsto \infty$, and using [10]

$$
\int_0^{\infty} dx \left(\frac{\sin x}{x}\right)^n \frac{\sin(mx)}{x} = \frac{\pi}{2} \quad m \geq n
$$

we can evaluate

$$
D^{(q.l.)} = \frac{\chi^2}{24} \qquad \text{(A24.58)}
$$

which is the correct result [5] (and again for cat maps (A24.58) is not the quasilinear estimate but the exact value of the diffusione coefficient). exercise A24.8

Commentary

Remark A24.1. Who has talked about it? Maps of this kind have been extensively analyzed as examples of genuine Hamiltonian chaotic evolution: in particular they admit simple Markov partitions [2, 9], which lead to simple analytic expressions for topological zeta functions [11]. The linear code was introduced by Percival and Vivaldi [4, 15]. Measure polytopes are discussed in ref. [7]. The quasilinear estimate (A24.50) was given in ref. [5]. (A24.50) was evaluated in ref. [3, 16]. Circulant matrix are discussed in ref. [1]. The result (A24.58) agrees with the saw-tooth result of ref. [5]; for the cat maps (A24.58) is the exact value of the diffusion coefficient. This result was obtained, by using periodic orbits also in ref. [8], where Gaussian nature of the diffusion process is explicitly assumed.

Remark A24.2. Discrete Fourier software. Wolfram Mathematica has an extensive and pedagogical suite of discrete Fourier transform modules.

Remark A24.3. Phase space. The cylinder phase is $[-1/2, 1/2)\times\mathbb{R}$: the map is originally defined on $[-1/2, 1/2)^2$, and is unfolded over the cylinder by symmetry (24.22).

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A24.6 Examples

Example A24.1. A 2-point lattice diagonalized.

The action of the stepping operator σ (A24.7) on a 2-point lattice $\phi = (\phi_0, \phi_1)$ is to permute the two lattice sites

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

As exchange repeated twice brings us back to the original state, $\sigma^2 = \mathbf{1}$, the characteristic polynomial of σ is

$$(\sigma + 1)(\sigma - 1) = 0 ,$$

with eigenvalues $\omega_0 = 1, \omega_1 = -1$. The symmetrization, antisymmetrization projection operators are

$$P_0 = \frac{\sigma - \omega_1 \mathbf{1}}{\omega_0 - \omega_1} = \frac{1}{2}(\mathbf{1} + \sigma) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{A24.59}$$

$$P_1 = \frac{\sigma - \mathbf{1}}{-1 - 1} = \frac{1}{2}(\mathbf{1} - \sigma) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} . \tag{A24.60}$$

Noting that $P_0 + P_1 = \mathbf{1}$, we can project a lattice state ϕ onto the two normalized eigenvectors of σ:

$$\phi = \mathbf{1}\,\phi = P_0 \cdot \phi + P_1 \cdot \phi ,$$

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \frac{(\phi_0 + \phi_1)}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{(\phi_0 - \phi_1)}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{A24.61}$$

$$= \tilde{\phi}_0\, \varphi_0 + \tilde{\phi}_1\, \varphi_1 . \tag{A24.62}$$

As $P_0 P_1 = 0$, the symmetric and the antisymmetric states transform separately under any linear transformation constructed from σ and its powers.

In this way the characteristic equation $\sigma^2 = \mathbf{1}$ enables us to reduce the 2-dimensional lattice state to two 1-dimensional ones, on which the value of the stepping operator σ is a number, $\omega_j \in \{1, -1\}$, and the normalized eigenvectors are $\varphi_0 = \frac{1}{\sqrt{2}}(1, 1)$, $\varphi_1 = \frac{1}{\sqrt{2}}(1, -1)$. As we shall now see, $(\tilde{\phi}_0, \tilde{\phi}_1)$ is the 2-site periodic lattice discrete Fourier transform of the field (ϕ_1, ϕ_2).

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Example A24.2. Projection operators for discrete Fourier transform / cyclic group C_N. (It's OK to skip this example on the first reading - the explicit Fourier eigenvectors and eigenvalues (A24.23) are all that we need to carry out discrete Fourier transforms.)

Consider a cyclic group

$$C_N = \{e, g, g^2, \cdots g^{N-1}\} , \quad g^N = e .$$

If $M = D(g)$ is a $[d \times d]$ matrix representation of the one-step shift g, it must satisfy $M^N - \mathbf{1} = 0$, with eigenvalues given by the zeros of the characteristic polynomial

$$G(x) = x^N - 1 = (x - \lambda_0)(x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_{N-1}) . \tag{A24.63}$$

For the cyclic group the N distinct eigenvalues are the Nth roots of unity $\lambda_n = \omega^n$, $\omega = \exp(i\,2\pi/N)$, $n = 0, \ldots N-1$.

In the projection operator formulation (A10.20), they split the d-dimensional space into d/N-dimensional subspaces by means of projection operators

$$P_n = \prod_{m\neq n} \frac{M - \omega^m I}{\omega^n - \omega^m} = \frac{1}{\prod_{m=1}^{N-1}(1-\omega^m)} \prod_{m=1}^{N-1} (\omega^{-n} M - \omega^m I)\,, \tag{A24.64}$$

where we have multiplied all denominators and numerators by ω^{-n}.

The denominator is a polynomial of form $G(x)/(x-\lambda_0)$, with the zeroth root $(x-\omega^0) = (x-1)$ quotiented out from the characteristic polynomial,

$$\frac{x^N - 1}{x - 1} = (x-\omega)(x-\omega^2)\cdots(x-\omega^{N-1})\,.$$

Consider a sum of the first N terms of a geometric series, multiplied by $(x-1)/(x-1)$:

$$1 + x + \cdots + x^{N-1} = \sum_{m=0}^{N-1} x^m = \frac{1}{x-1} \sum_{m=0}^{N-1} (x-1)\, x^m = \frac{x^N - 1}{x - 1}\,. \tag{A24.65}$$

So, the products in (A24.64) can be written as sums

$$(x-\omega)(x-\omega^2)\cdots(x-\omega^{N-1}) = 1 + x + \cdots + x^{N-1}\,. \tag{A24.66}$$

The P_n projection operator (A24.64) denominator is evaluated by substituting $x \to 1$ into (A24.66); that adds up to N. The numerator is evaluated by substituting $x \to \omega^{-n} M$. We obtain the projection operator as a discrete Fourier weighted sum of matrices M^m,

$$P_n = \frac{1}{N} \sum_{m=0}^{N-1} e^{-i\,\frac{2\pi}{N} nm}\, M^m\,, \tag{A24.67}$$

instead of the product form (A24.64).

This is the simplest example of the key group theory tool, the projection operator expressed as a sum over characters,

$$P_n = \frac{1}{|G|} \sum_{g\in G} \bar\chi_n(g) D(g)\,.$$

As C_N irreps are all 1-dimensional, for the discrete Fourier transform all characters are simply $\bar\chi_n(g^m) = \omega^{-nm}$, the Nth complex roots of unity.

(B. Gutkin and P. Cvitanović)

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Example A24.3. 'Configuration-momentum' Fourier space duality.

What does a projection on the kth Fourier subspace mean? The discrete Fourier transform (A24.67) of a state (vector) ϕ rearranges components of vector ϕ into averages over all its components, weighted by complex phases $\exp(i2\pi\ell/N)$ in all possible ways.

Consider first the projection on the 0th Fourier mode

$$P_0 = \frac{1}{N} \sum_{m=0}^{N-1} M^m\,.$$

Applied to a lattice state $\phi = (\phi_1, \phi_2, \cdots, \phi_N)$, the shift matrix M translates the state by one site, $M\phi = (\phi_2, \phi_3, \cdots, \phi_N, \phi_1)$, and so on for all powers M^m. The result is the space average (here correctly normalized, so that $\langle 1 \rangle = 1$) over all values of the periodic lattice field ϕ_m,

$$\frac{1}{\sqrt{N}}\tilde{\phi}_0 = \frac{1}{N}\sum_{\ell=0}^{N-1}\phi_\ell = \langle\phi\rangle \,,$$

see (A24.17) and (A24.26). Every finite discrete group has such fully-symmetric representation, and in statistical mechanics and quantum mechanics this is often the most important state (the 'ground' state).

$\tilde{\phi}_1$ is the average weighted by one oscillation over the N-periodic lattice, and $\tilde{\phi}_k$, the projection of the N-dimensional state (i.e., vector) ϕ on the kth subspace

$$\tilde{\phi}_k = P_k \cdot \phi = \frac{1}{\sqrt{N}}\sum_{\ell=0}^{N-1} e^{-i\frac{2\pi}{N}k\ell}\phi_\ell \,, \tag{A24.68}$$

is the average weighted by complex rotating phase ω^{km} which advances by ω^k in every step, and pulls out oscillating feature $\tilde{\phi}_k$ out of the field ϕ. For large N, modes $\tilde{\phi}_k$ with $k \ll N$ (or $(N-k) \ll N$, that is just a counter-rotation)) are called *hydrodynamic modes*, corresponding to "configuration" lattice fields ϕ which vary slowly and smoothly over many lattice spacings. Modes with $k \simeq N/2$ are suspect, they are lattice discretization artifacts.

If the lattice state is ϕ is localized, its Fourier transform will be global, and vice versa for a localized Fourier state $\tilde{\phi}$. For example, if the field ϕ is concentrated on the first site, $\phi_0 = 1$, rest zero, it's Fourier transform will be uniformly distributed over all Fourier modes, $\tilde{\phi}_k = 1/\sqrt{N}$.

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Exercises

A24.1. **Laplacian is a non-local operator.**
While the Laplacian is a simple tri-diagonal difference operator (A24.10), its inverse (the "free" propagator of statistical mechanics and quantum field theory) is a messier object. A way to compute is to start expanding propagator as a power series in the Laplacian

$$\frac{1}{m^2\mathbf{1}-\Box} = \frac{1}{m^2}\sum_{n=0}^{\infty}\frac{1}{m^{2n}}\Box^n\,. \tag{A24.69}$$

As $\Box$ is a finite matrix, the expansion is convergent for sufficiently large m^2. To get a feeling for what is involved in evaluating such series, show that $\Box^2$ is:

$$\Box^2 = \frac{1}{a^4}\left[\begin{array}{ccccccc} 6 & -4 & 1 & & & 1 & -4 \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & & & & & & \\ & 1 & -4 & \ddots & & & \\ & & & & & 6 & -4 \\ -4 & 1 & & & 1 & -4 & 6 \end{array}\right]. \tag{A24.70}$$

What $\Box^3$, $\Box^4$, $\cdots$ contributions look like is now clear; as we include higher and higher powers of the Laplacian, the propagator matrix fills up; while the *inverse* propagator is differential operator connecting only the nearest neighbors, the propagator is integral operator, connecting every lattice site to any other lattice site.

This matrix can be evaluated as is, on the lattice, and sometime it is evaluated this way, but in case at hand a wonderful simplification follows from the observation that the lattice action is translationally invariant, exercise A24.2.

A24.2. **Lattice Laplacian diagonalized.** Insert the identity $\sum P^{(k)} = \mathbf{1}$ wherever you profitably can, and use the eigenvalue equation (A24.23) to convert shift σ matrices into scalars. If M *commutes* with σ, then $(\varphi_k^\dagger\cdot M\cdot\varphi_{k'}) = \tilde{M}^{(k)}\delta_{kk'}$, and the matrix M acts as a multiplication by the scalar $\tilde{M}^{(k)}$ on the kth subspace. Show that for the 1-dimensional version of the lattice Laplacian (A24.10) the projection on the kth subspace is

$$(\varphi_k^\dagger\cdot\Box\cdot\varphi_{k'}) = \frac{2}{a^2}\left(\cos\left(\frac{2\pi}{N}k\right)-1\right)\delta_{kk'}\,. \tag{A24.71}$$

In the kth subspace the propagator is simply a number, and, in contrast to the mess generated by (A24.69), there is nothing to evaluating:

$$\varphi_k^\dagger\cdot\frac{1}{m^2\mathbf{1}-\Box}\cdot\varphi_{k'} = \frac{\delta_{kk'}}{m^2-\frac{2}{(ma)^2}(\cos 2\pi k/N-1)}\,, \tag{A24.72}$$

where k is a site in the N-dimensional dual lattice, and $a = L/N$ is the lattice spacing.

A24.3. **Recursion relations.** Verify that the following recursion relations are satisfied

$$u_{k+2} = \chi u_{k+1} - u_k$$

where $u_k = A_k, B_k, C_k, D_k$.

A24.4. **Arnol'd cat map.** Show that for $\chi = 3$, $A_k = F_{2k+1}$, $B_k = L_{2k+1}$, $C_k = L_{2k}$ and $D_k = 5F_{2k}$, where F_n and L_n are the Fibonacci and Lucas numbers..

A24.5. **Pruning rules for substrings of length 2.** Take $K = 8$ and draw the region determined by (A24.44).

A24.6. **Diagonalization of A.** Show that A can be diagonalized by considering the auxiliary matrix U

$$U \overset{\text{def}}{=} \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \epsilon_0 & \epsilon_1 & \cdots & \epsilon_{T-2} & \epsilon_{T-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \epsilon_0^{T-2} & \epsilon_1^{T-2} & \cdots & \epsilon_{T-2}^{T-2} & \epsilon_{T-1}^{T-2} \\ \epsilon_0^{T-1} & \epsilon_1^{T-1} & \cdots & \epsilon_{T-2}^{T-1} & \epsilon_{T-1}^{T-1} \end{pmatrix}.$$

In fact $U^{-1}AU$ is a diagonal matrix (the diagonal elements coinciding with $f(\epsilon_j)$).

A24.7. **Periodic points of cat maps.** Verify that (A24.55) is exactly the number of T–periodic points of the map when K is an integer.

A24.8. **Probability distribution.** Higher order moments can be computed easily for integer K (or generic K within the quasilinear approximation), by generalizations of (A24.57): show that the results prove that, given a period T, the distribution of periodic orbits with respect to their jumping number is asymptotically Gaussian, with parameter $D^{(q.l.)}$.

A24.9. **Deterministic diffusion, zig-zag map.**
To illustrate the main idea of chapter 24, tracking of a globally diffusing orbit by the associated confined orbit restricted to the fundamental cell, we consider a class of simple 1-dimensional dynamical systems, chains of piecewise linear maps, where all transport coefficients can be evaluated analytically. The translational symmetry (24.22) relates the unbounded dynamics on the real line to the dynamics restricted to a "fundamental cell" -

in the present example the unit interval curled up into a circle. An example of such map is the sawtooth map

$$\hat{f}(x) = \begin{cases} \Lambda x & x \in [0, 1/4 + 1/4\Lambda] \\ -\Lambda x + (\Lambda + 1)/2 & x \in [1/4 + 1/4\Lambda, 3/4 - 1/4\Lambda] \\ \Lambda x + (1 - \Lambda) & x \in [3/4 - 1/4\Lambda, 1] \end{cases} . \tag{A24.73}$$

The corresponding circle map $f(x)$ is obtained by modulo the integer part. The elementary cell map $f(x)$ is sketched in figure **??**. The map has the symmetry property

$$\hat{f}(\hat{x}) = -\hat{f}(-\hat{x}), \tag{A24.74}$$

so that the dynamics has no drift, and all odd derivatives of the generating function (24.4) with respect to β evaluated at $\beta = 0$ vanish.

The cycle weights are given by

$$t_p = z^{n_p} \frac{e^{\beta \hat{n}_p}}{|\Lambda_p|} . \tag{A24.75}$$

The diffusion constant formula for 1-dimensional maps is

$$D = \frac{1}{2} \frac{\langle \hat{n}^2 \rangle_\zeta}{\langle n \rangle_\zeta} \tag{A24.76}$$

where the "mean cycle time" is given by

$$\langle n \rangle_\zeta = z \frac{\partial}{\partial z} \frac{1}{\zeta(0, z)} \bigg|_{z=1} = -\sum{}' (-1)^k \frac{n_{p_1} + \cdots + n_{p_k}}{|\Lambda_{p_1} \cdots \Lambda_{p_k}|} , \tag{A24.77}$$

the mean cycle displacement squared by

$$\langle \hat{n}^2 \rangle_\zeta = \frac{\partial^2}{\partial \beta^2} \frac{1}{\zeta(\beta, 1)} \bigg|_{\beta=0} = -\sum{}' (-1)^k \frac{(\hat{n}_{p_1} + \cdots + \hat{n}_{p_k})^2}{|\Lambda_{p_1} \cdots \Lambda_{p_k}|} , \tag{A24.78}$$

and the sum is over all distinct non-repeating combinations of prime cycles. Most of results expected in this projects require no more than pencil and paper computations.

Implementing the symmetry factorization (24.19) is convenient, but not essential for this project, so if you find example 25.9 too long a read, skip the symmetrization.

A24.10. **The full shift sawtooth map.** Take the map (A24.73) and extend it to the real line. As in example of figure 24.4, denote by a the critical value of the map (the maximum height in the unit cell)

$$a = \hat{f}(\frac{1}{4} + \frac{1}{4\Lambda}) = \frac{\Lambda + 1}{4} . \tag{A24.79}$$

Describe the symbolic dynamics that you obtain when a is an integer, and derive the formula for the diffusion constant:

$$D = \frac{(\Lambda^2 - 1)(\Lambda - 3)}{96\Lambda} \qquad \text{for } \Lambda = 4a - 1, \ a \in \mathbb{Z} . \tag{A24.80}$$

If you are going strong, derive also the fromula for the half-integer $a = (2k + 1)/2$, $\Lambda = 4a + 1$ case and email it to predrag@nbi.dk. You will need to partition $\mathcal{M}_2$ into the left and right half, $\mathcal{M}_2 = \mathcal{M}_8 \cup \mathcal{M}_9$, as in the derivation of (24.29). See exercise 24.1.

A24.11. **Sawtooth map subshifts of finite type.** We now work out an example when the partition is Markov, although the slope is not an integer number. The key step is that of having a partition where intervals are mapped *onto* unions of intervals. Consider for example the case in which $\Lambda = 4a - 1$, where $1 \leq a \leq 2$. A first partition is constructed from seven intervals, which we label $\{\mathcal{M}_1, \mathcal{M}_4, \mathcal{M}_5, \mathcal{M}_2, \mathcal{M}_6, \mathcal{M}_7, \mathcal{M}_3\}$, with the alphabet ordered as the intervals are laid out along the unit interval. In general the critical value a will not correspond to an interval border, but now we choose a such that the critical point is mapped onto the right border of $\mathcal{M}_1$, as in figure **??** (a). The critical value of $f()$ is $f(\frac{\Lambda+1}{4\Lambda}) = a - 1 = (\Lambda - 3)/4$. Equating this with the right border of $\mathcal{M}_1$, $x = 1/\Lambda$, we obtain a quadratic equation with the expanding solution $\Lambda = 4$. We have that $f(\mathcal{M}_4) = f(\mathcal{M}_5) = \mathcal{M}_1$, so the transition matrix (17.1) is given by

$$\phi' = T\phi = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_4 \\ \phi_5 \\ \phi_2 \\ \phi_6 \\ \phi_7 \\ \phi_3 \end{bmatrix} \tag{A24.81}$$

and the dynamics is unrestricted in the alphabet

$$\{1, \underline{41}, \underline{51}, 2, \underline{63}, \underline{73}, 3, \} .$$

One could diagonalize (A24.81) on the computer, but, as we saw in chapter 17, the transition graph figure **??** (b) corresponding to figure **??** (a) offers more insight into the dynamics. The dynamical zeta function

$$\begin{aligned} 1/\zeta &= 1 - (t_1 + t_2 + t_3) - 2(t_{14} + t_{37}) \\ 1/\zeta &= 1 - 3\frac{z}{\Lambda} - 4\cosh\beta \frac{z^2}{\Lambda^2} . \end{aligned} \tag{A24.82}$$

follows from the loop expansion (18.13) of sect. 18.3.

The material flow conservation sect. 23.4 and the symmetry factorization (24.19) yield

$$0 = \frac{1}{\zeta(0, 1)} = \left(1 + \frac{1}{\Lambda}\right)\left(1 - \frac{4}{\Lambda}\right)$$

which indeed is satisfied by the given value of Λ. Conversely, we can use the desired Markov partition topology to write down the corresponding dynamical zeta function, and use the $1/\zeta(0,1) = 0$ condition to fix Λ. For more complicated transition matrices the factorization (24.19) is very helpful in reducing the order of the polynomial condition that fixes Λ.

The diffusion constant follows from (24.20) and (A24.76)

$$\langle n \rangle_\zeta = -\left(1+\frac{1}{\Lambda}\right)\left(-\frac{4}{\Lambda}\right), \quad \langle \hat{n}^2 \rangle_\zeta = \frac{4}{\Lambda^2}$$

$$D = \frac{1}{2}\frac{1}{\Lambda+1} = \frac{1}{10}$$

Think up other non-integer values of the parameter for which the symbolic dynamics is given in terms of Markov partitions: in particular consider the cases illustrated in figure **??** and determine for what value of the parameter a each of them is realized. Work out the transition graph, symmetrization factorization and the diffusion constant, and check the material flow conservation for each case. Derive the diffusion constants listed in table **??**. It is not clear why the final answers tend to be so simple. Numerically, the case of figure **??** (c) appears to yield the maximal diffusion constant. Does it? Is there an argument that it should be so?

The seven cases considered here (see table **??**, figure **??** and (A24.80)) are the 7 simplest complete Markov partitions, the criterion being that the critical points map onto partition boundary points. This is, for example, what happens for unimodal tent map; if the critical point is preperiodic to an unstable cycle, the grammar is complete. The simplest example is the case in which the tent map critical point is preperiodic to a unimodal map 3-cycle, in which case the grammar is of golden mean type, with _00_ substring prohibited (see figure 17.7). In case at hand, the "critical" point is the junction of branches 4 and 5 (symmetry automatically takes care of the other critical point, at the junction of branches 6 and 7), and for the cases considered the critical point maps into the endpoint of each of the seven branches.

One can fill out parameter a axis arbitrarily densely with such points - each of the 7 primary intervals can be subdivided into 7 intervals obtained by 2-nd iterate of the map, and for the critical point mapping into any of those in 2 steps the grammar (and the corresponding cycle expansion) is finite, and so on.

A24.12. **Sawtooth map diffusion coefficient, numerically.** *(optional:)*
Attempt a numerical evaluation of

$$D = \frac{1}{2}\lim_{n\to\infty}\frac{1}{n}\langle \hat{x}_n^2 \rangle . \tag{A24.83}$$

Study the convergence by comparing your numerical results to the exact answers derived above. Is it better to use few initial $\hat{x}$ and average for long times, or to use many initial $\hat{x}$ for shorter times? Or should one fit the distribution of $\hat{x}^2$ with a Gaussian and get the D this way? Try to plot dependence of D on Λ; perhaps blow up a small region to show that the dependance of D on the parameter Λ is fractal. Compare with figure 24.6 and figures in refs. [6, 12–14, 17].

A24.13. **Sawtooth D is a nonuniform function of the parameters.** *(optional:)*
The dependence of D on the map parameter Λ is rather unexpected - even though for larger Λ more points are mapped outside the unit cell in one iteration, the diffusion constant does not necessarily grow. An interpretation of this lack of monotonicity would be interesting.

You can also try applying periodic orbit theory to the sawtooth map (A24.73) for a random "generic" value of the parameter Λ, for example $\Lambda = 6$. The idea is to bracket this value of Λ by the nearby ones, for which higher and higher iterates of the critical value $a = (\Lambda+1)/4$ fall onto the partition boundaries, compute the exact diffusion constant for each such approximate Markov partition, and study their convergence toward the value of D for $\Lambda = 6$. Judging how difficult such problem is already for a tent map (see sect. 18.5 and appendix A18.2), this is too ambitious for a week-long exam.

A24.14. **Deterministic diffusion, sawtooth map.**

To illustrate the main idea of chapter 24, tracking of a globally diffusing orbit by the associated confined orbit restricted to the fundamental cell, we consider in more detail the class of simple 1-dimensional dynamical systems, chains of piecewise linear maps (24.21). The translational symmetry (24.22) relates the unbounded dynamics on the real line to the dynamics restricted to a "fundamental cell" - in the present example the unit interval curled up into a circle. The corresponding circle map $f(x)$ is obtained by modulo the integer part. The elementary cell map $f(x)$ is sketched in figure 24.4. The map has the symmetry property

$$\hat{f}(\hat{x}) = -\hat{f}(-\hat{x}), \tag{A24.84}$$

so that the dynamics has no drift, and all odd derivatives of the generating function (24.4) with respect to β evaluated at $\beta = 0$ vanish.

The cycle weights are given by

$$t_p = z^{n_p}\frac{e^{\beta \hat{n}_p}}{|\Lambda_p|}. \tag{A24.85}$$

The diffusion constant formula for 1-dimensional maps is

$$D = \frac{1}{2}\frac{\langle \hat{n}^2 \rangle_\zeta}{\langle n \rangle_\zeta} \tag{A24.86}$$

where the "mean cycle time" is given by

$$\langle n \rangle_\zeta = z\frac{\partial}{\partial z}\frac{1}{\zeta(0,z)}\bigg|_{z=1} = -\sum{}'(-1)^k \frac{n_{p_1} + \cdots + n_{p_k}}{|\Lambda_{p_1}\cdots\Lambda_{p_k}|}\,, \tag{A24.87}$$

the mean cycle displacement squared by

$$\langle \hat{n}^2 \rangle_\zeta = \frac{\partial^2}{\partial \beta^2}\frac{1}{\zeta(\beta,1)}\bigg|_{\beta=0} = -\sum{}'(-1)^k \frac{(\hat{n}_{p_1} + \cdots + \hat{n}_{p_k})^2}{|\Lambda_{p_1}\cdots\Lambda_{p_k}|}\,, \tag{A24.88}$$

and the sum is over all distinct non-repeating combinations of prime cycles. Most of results expected in this projects require no more than pencil and paper computations.