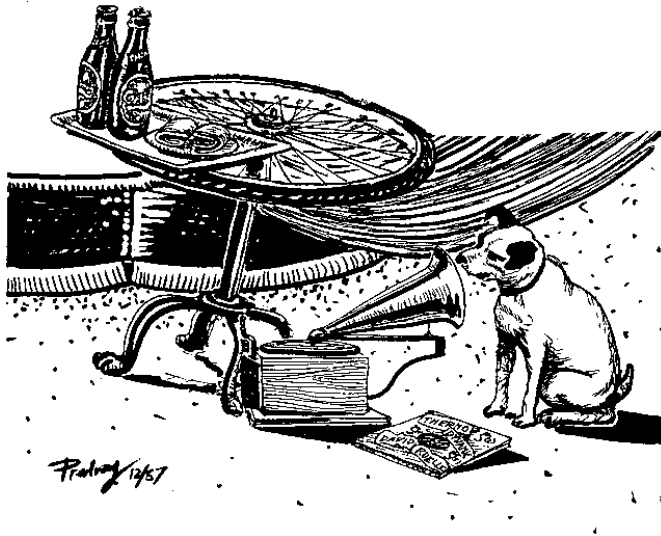


Chaos: Classical and Quantum

I: Deterministic Chaos



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No man but a blockhead ever wrote except for money
—Samuel Johnson

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Daniel Borrero Oct 23 2008, soluCycles.tex

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