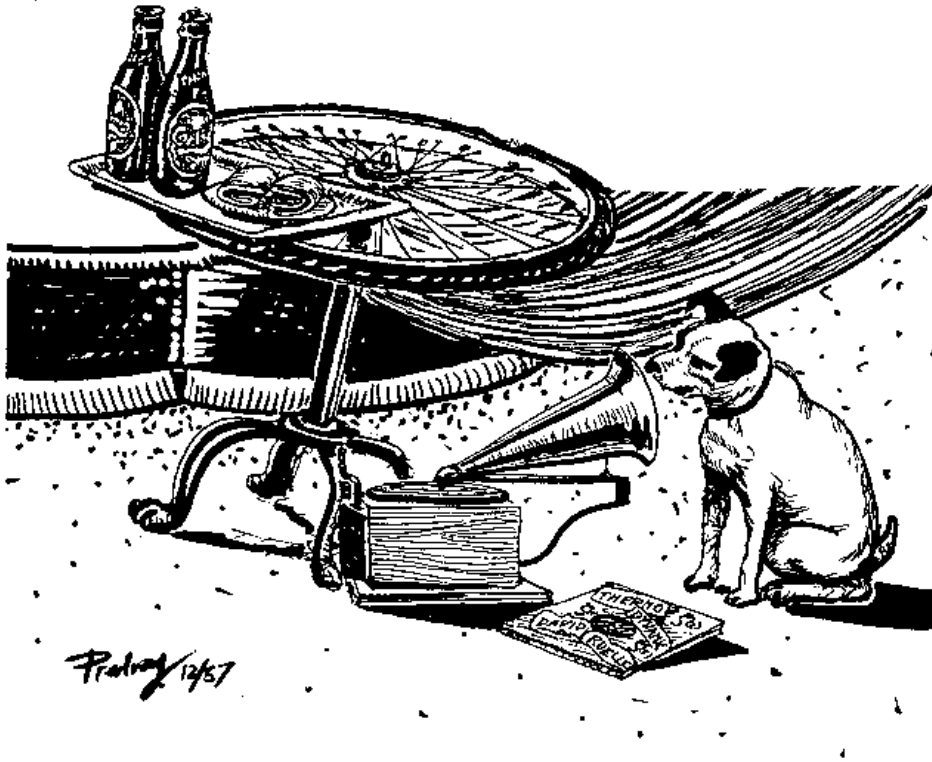


Chaos: Classical and Quantum



Predrag Cvitanović – Roberto Artuso – Ronnie Mainieri – Gregor Tanner –
Gábor Vattay

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Acknowledgments

I feel I never want to write another book. What's the good!
I can eke living on stories and little articles, that don't cost
a tith of the output a book costs. Why write novels any
more!

—D.H. Lawrence

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F. Haake's heartfelt lament on page 390 was uttered at the end of the first conference presentation of cycle expansions, in 1988. G.P. Morriss advice to students as how to read the introduction to this book, page 5, was offered during a 2002 graduate course in Dresden. J. Bellissard's advice to students concerning unpleasant operators and things nonlinear, pages 4.3 and 20.3.1, was shared in his 2013 "Classical Mechanics II" lectures on manifolds. K. Huang's C.N. Yang interview quoted on page 355 is available on ChaosBook.org/extras. T.D. Lee

remarks on as to who is to blame, page 36 and page 294, as well as M. Shub's helpful technical remark on page 540 came during the Rockefeller University December 2004 "Feigenbaum Fest." Quotes on pages 36, 137, and 352 are taken from a book review by J. Guckenheimer [1].

Who is the 3-legged dog reappearing throughout the book? Long ago, when we were innocent and knew not Borel measurable α to Ω sets, P. Cvitanović asked V. Baladi a question about dynamical zeta functions, who then asked J.-P. Eckmann, who then asked D. Ruelle. The answer was transmitted back: "The master says: 'It is holomorphic in a strip'." Hence His Master's Voice logo, and the 3-legged dog is us, still eager to fetch the bone. The answer has made it to the book, though not precisely in His Master's voice. As a matter of fact, the answer *is* the book. We are still chewing on it.

What about the two beers? During his PhD studies, R. Artuso found the smørrebrød at the Niels Bohr Institute indigestible, so he digested H.M.V.'s wisdom on a strict diet of two Carlsbergs and two pieces of danish pastry for lunch every day, as depicted on the cover. Frequent trips back to Milano family kept him alive—he never got desperate enough to try the Danish smørrebrød. And the cycle wheel? Well, this is no book for pedestrians.

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