Chapter 25

Discrete symmetry factorization

> No endeavor that is worthwhile is simple in prospect; if it is right, it will be simple in retrospect.
>
> —Edward Teller

To those versed in Quantum Mechanics (QM), utility of symmetries in reducing spectrum calculations is *sine qua non*: if a group of symmetries commutes with the Hamiltonian, irreducible representations of the symmetry group block-diagonalize it, each block spanned by a set of the degenerate eigenstates of the same energy. Like most QM gymnastics, this block-diagonalization has nothing to do with quantum mysteries, it is just linear algebra. As we shall show here, classical spectral determinants factor in the same way, given that the evolution operator $\mathcal{L}^t(y, x)$ for a system $f^t(x)$ is invariant under a discrete symmetry group $G = \{e, g_2, g_3, \cdots, g_{|G|}\}$ of order $|G|$. In the process we 1.) learn that the classical dynamics, once recast into the language of evolution operators, is much closer to quantum mechanics than is apparent in the Newtonian, ODE formulation (linear evolution operators, group-theoretical spectral decompositions, . . .), 2.) that once the symmetry group is quotiented out, the dynamics simplifies, and 3.) it's a triple home run: simpler symbolic dynamics, fewer cycles needed, much better convergence of cycle expansions. Once you master this, going back to your pre-desymmetrization ways is unthinkable.

The main result of this chapter can be stated as follows:

If the dynamics possesses a discrete symmetry, the contribution of a cycle p of multiplicity m_p to a dynamical zeta function factorizes into a product over the d_μ-dimensional irreps $D^{(\mu)}(g)$ of the symmetry group,

$$(1 - t_p)^{m_p} = \prod_\mu \det\left(1 - D^{(\mu)}(h_{\hat{p}})t_{\hat{p}}\right)^{d_\mu}, \quad t_p = t_{\hat{p}}^{|G|/m_p},$$

where $t_{\hat{p}}$ is the cycle weight evaluated on the relative periodic orbit $\hat{p}$, $|G|$ is the order of the group, $h_{\hat{p}}$ is the group element relating the fundamental domain cycle $\hat{p}$ to a segment of the full space cycle p, and m_p is the multiplicity of the p cycle.

As dynamical zeta functions have particularly simple cycle expansions, a geometrical shadowing interpretation of their convergence, and suffice for determination of leading eigenvalues, we shall use them to explain the group-theoretic factorizations; the full spectral determinants can be factorized using the same techniques.

This chapter is meant to serve as a detailed guide to the computation of dynamical zeta functions and spectral determinants for systems with discrete symmetries. Familiarity with basic group-theoretic notions is assumed, with some details relegated to appendix A10.1. We develop here the cycle expansions for factorized determinants, and exemplify them by working out two cases of physical interest: $C_2 = D_1$ and $C_{3v} = D_3$ symmetries. $C_{2v} = D_1 \times D_1$ and $C_{4v} = D_4$ symmetries are discussed in appendix A25. We start with a review of some basic facts of the group representation theory.

25.1 Transformation of functions

So far we have recast the problem of long time dynamics into language of linear operators acting of functions, simplest one of which is $\rho(x,t)$, the density of trajectories at time t. First we will explain what discrete symmetries do to such functions, and then how they affect their evolution in time.

Let g be an *abstract group element* in G. For a discrete group a group element is typically indexed by a discrete label, $g = g_j$. For a continuous group it is typically parametrized by a set of continuous parameters, $g = g(\theta_m)$. As discussed on page 167, linear action of a group element $g \in G$ on a state $x \in \mathcal{M}$ is given by its *matrix representation*, a finite non-singular $[d \times d]$ matrix $D(g)$:

$$x \to x' = D(g)\, x \,. \tag{25.1}$$

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How does the group act on a function ρ of x? Denote by $U(g)$ the operator $\rho'(x) = U(g)\,\rho(x)$ that returns the transformed function. One *defines* the transformed function ρ' by requiring that it has the same value at $x' = D(g)x$ as the initial function has at x,

$$\rho'(x') = U(g)\,\rho(D(g)x) = \rho(x) \,.$$

Replacing $x \to D(g)^{-1}x$, we find that a group element $g \in G$ acts on a function $\rho(x)$ defined on state space $\mathcal{M}$ by its *operator representation*

$$U(g)\,\rho(x) = \rho(D(g)^{-1}x) \,. \tag{25.2}$$

This is the conventional, Wigner definition of the effect of transformations on functions that should be familiar to master quantum mechanicians. Again: $U(g)$ is an *'operator'*, not a matrix - it is an operation whose only meaning is exactly what

(25.2) says. And yes, Mathilde, the action on the state space points is $D(g)^{-1}x$, not $D(g)x$.

Consider next the effect of two successive transformations g_1, g_2:

$$\begin{aligned} U(g_2)U(g_1)\rho(x) &= U(g_2)\rho(D(g_1)^{-1}x) = \rho(D(g_2)^{-1}D(g_1)^{-1}x) \\ &= \rho(D(g_1g_2)^{-1}x) = U(g)\rho(x)\,. \end{aligned}$$

Hence if $g_1g_2 = g$, we have $U(g_2)U(g_1) = U(g)$: so operators $U(g)$ form a representation of the group.

25.2 Taking care of fundamentals

> Instant gratification takes too long.
> — Carrie Fisher

If a dynamical system $(\mathcal{M}, f)$ is equivariant under a discrete symmetry (visualize the 3-disk billiard, figure 11.2), the state space $\mathcal{M}$ can be tiled by a *fundamental domain* $\hat{\mathcal{M}}$ and its images $\hat{\mathcal{M}}_2 = g_2\hat{\mathcal{M}}$, $\hat{\mathcal{M}}_3 = g_3\hat{\mathcal{M}}$, ... under the action of the symmetry group $G = \{e, g_2, \dots, g_{|G|}\}$,

section 11.3

$$\mathcal{M} = \sum_{g\in G} \hat{\mathcal{M}}_g = \hat{\mathcal{M}} \cup \hat{\mathcal{M}}_2 \cup \hat{\mathcal{M}}_3 \cdots \cup \hat{\mathcal{M}}_{|G|}\,. \tag{25.3}$$

example 25.3
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25.2.1 Regular representation

Take an arbitrary function $\rho(x)$ defined over the state space $x \in \mathcal{M}$. If the state space is tiled by a fundamental domain $\hat{\mathcal{M}}$ and its copies, function $\rho(x)$ can be written as a $|G|$-dimensional vector of functions, each function defined over the fundamental domain $\hat{x} \in \hat{\mathcal{M}}$ only. The natural choice of a function space basis is the $|G|$-component *regular basis* vector

$$\begin{bmatrix} \rho_1^{reg}(\hat{x}) \\ \rho_2^{reg}(\hat{x}) \\ \vdots \\ \rho_{|G|}^{reg}(\hat{x}) \end{bmatrix} = \begin{bmatrix} \rho(\,D(e)\hat{x}\,) \\ \rho(D(g_2)\hat{x}) \\ \vdots \\ \rho(D(g_{|G|})\hat{x}) \end{bmatrix}, \tag{25.4}$$

constructed from an arbitrary function $\rho(x)$ defined over the entire state space $\mathcal{M}$, by applying $U(g^{-1})$ to $\rho(\hat{x})$ for each $g \in G$, with state space points restricted to the fundamental domain, $\hat{x} \in \hat{\mathcal{M}}$.

Now apply group action *operator* $U(g)$ to a regular basis vector:

$$U(g)\begin{bmatrix}\rho(\,D(e)\hat{x}\,)\\ \rho(D(g_2)\hat{x})\\ \vdots\\ \rho(D(g_{|G|})\hat{x})\end{bmatrix} = \begin{bmatrix}\rho(\,D(g^{-1})\hat{x}\,)\\ \rho(D(g^{-1}g_2)\hat{x})\\ \vdots\\ \rho(D(g^{-1}g_{|G|})\hat{x})\end{bmatrix}.$$

It acts by permuting the components. (And yes, Mathilde, the pesky g^{-1} is inherited from (25.2), and there is nothing you can do about it.) Thus the action of the *operator* $U(g)$ on a regular basis vector can be represented by the corresponding $[|G|\times|G|]$ permutation *matrix*, called the *left regular representation* $D^{reg}(g)$,

$$U(g)\begin{bmatrix}\rho_1^{reg}(\hat{x})\\ \rho_2^{reg}(\hat{x})\\ \vdots\\ \rho_{|G|}^{reg}(\hat{x})\end{bmatrix} = D^{reg}(g)\begin{bmatrix}\rho_1^{reg}(\hat{x})\\ \rho_2^{reg}(\hat{x})\\ \vdots\\ \rho_{|G|}^{reg}(\hat{x})\end{bmatrix}.$$

A product of two permutations is a permutation, so this is a matrix representation of the group. To compute its entries, write out the matrix multiplication explicitly, labeling the vector components by the corresponding group elements,

$$\rho_b^{reg}(\hat{x}) = \sum_a^G D^{reg}(g)_{ba}\,\rho_a^{reg}(\hat{x})\,.$$

A product of two group elements $g^{-1}a$ is a unique element b, so the a_{th} row of $D^{reg}(g)$ is all zeros, except the b_{th} column which satisfies $g = b^{-1}a$. We arrange the columns of the multiplication table by the inverse group elements, as in table 25.1. Setting multiplication table entries with g to 1, and the rest to 0 then defines the regular representation *matrix* $D^{reg}(g)$ for a given g,

$$D^{reg}(g)_{ab} = \delta_{g,b^{-1}a}\,. \tag{25.5}$$

For instance, in the case of the 2-element group $\{e,\sigma\}$ the $D^{reg}(g)$ can be either the identity or the interchange of the two domain labels,

$$D^{reg}(e) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\,,\quad D^{reg}(\sigma) = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\,. \tag{25.6}$$

The multiplication table for D_3 is a more typical, nonabelian group example: see table 25.1. The multiplication tables for C_2 and C_3 are given in table 25.2.

The regular representation of group identity element e is always the identity matrix. As $D^{reg}(g)$ is a permutation matrix, mapping a tile $\hat{\mathcal{M}}_a$ into a different tile $\hat{\mathcal{M}}_{ga} \neq \hat{\mathcal{M}}_a$ if $g \neq e$, only $D^{reg}(e)$ has diagonal elements, and

$$\mathrm{tr}\,D^{reg}(g) = |G|\,\delta_{g,e}\,. \tag{25.7}$$

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D_3	e	σ_{12}	σ_{23}	σ_{31}	$C^{1/3}$	$C^{2/3}$
e	e	σ_{12}	σ_{23}	σ_{31}	$C^{1/3}$	$C^{2/3}$
$(\sigma_{12})^{-1}$	σ_{12}	e	$C^{1/3}$	$C^{2/3}$	σ_{23}	σ_{31}
$(\sigma_{23})^{-1}$	σ_{23}	$C^{2/3}$	e	$C^{1/3}$	σ_{31}	σ_{12}
$(\sigma_{31})^{-1}$	σ_{31}	$C^{1/3}$	$C^{2/3}$	e	σ_{12}	σ_{23}
$(C^{1/3})^{-1}$	$C^{2/3}$	σ_{23}	σ_{31}	σ_{12}	e	$C^{1/3}$
$(C^{2/3})^{-1}$	$C^{1/3}$	σ_{31}	σ_{12}	σ_{23}	$C^{2/3}$	e

$$D^{reg}(\sigma_{23}) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad D^{reg}(C^{1/3}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Table 25.1: (top) The multiplication table of D_3, the group of symmetries of a triangle. (bottom) By (25.5), the 6 regular representation matrices $D^{reg}(g)$ of dihedral group D_3 have '1' at the location of g in the D_3 multiplication table table 25.1, '0' elsewhere. For example, the regular representation of the action of operators $U(\sigma_{23})$ and $U(C^{2/3})$ on the regular basis (25.4) are shown here.

25.2.2 Irreps: to get invariants, average

A representation $D^{(\mu)}(g)$ acting on d_μ-dimensional vector space $V^{(\mu)}$ is an *irreducible representation* (*irrep*) of group G if its only invariant subspaces are $V^{(\mu)}$ and the null vector $\{0\}$. To develop a feeling for this, one can train on a number of simple examples, and work out in each case explicitly a similarity transformation S that brings $D^{reg}(g)$ to a block diagonal form

$$S^{-1} D^{reg}(g) S = \begin{bmatrix} D^{(1)}(g) & & \\ & D^{(2)}(g) & \\ & & \ddots \end{bmatrix} \tag{25.8}$$

for every group element g, such that the corresponding subspace is invariant under actions $g \in G$, and contains no further nontrivial subspace within it. For the problem at hand we do not need to construct invariant subspaces $\rho^{(\mu)}(x)$ and $D^{(\mu)}(g)$ explicitly. We are interested in the symmetry reduction of the trace formula, and for that we will need only one simple result (lemma, theorem, whatever): the regular representation of a finite group contains all of its irreps μ, and its trace is given by the sum

$$\mathrm{tr}\, D^{reg}(g) = \sum_\mu d_\mu \chi^{(\mu)}(g)\,, \tag{25.9}$$

where d_μ is the dimension of irrep μ, and the characters $\chi^{(\mu)}(g)$ are numbers *intrinsic* to the group G that have to be tabulated only once in the history of humanity. And they all have been. The finiteness of the number of irreps and their dimensions d_μ follows from the dimension sum rule for $\mathrm{tr}\, D^{reg}(e)$, $|G| = \sum d_\mu^2$.

The simplest example is afforded by the 1-dimensional subspace (irrep) given by the fully symmetrized average of components of the regular basis function

$\rho^{reg}(x)$

$$\rho^{(A_1)}(x) = \frac{1}{|G|}\sum_g^G \rho(D(g)\,x)\,.$$

By construction, $\rho^{(A_1)}$ is invariant under all actions of the group, $U(g)\rho^{(A_1)}(x) = \rho^{(A_1)}(x)$. In other words, for every g this is an eigenvector of the regular representation $D^{reg}(g)$ with eigenvalue 1. Other eigenvalues, eigenvectors follow by working out C_3, C_N (discrete Fourier transform!) and D_3 examples.

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The beautiful Frobenius 'character orthogonality' theory of irreps (irreducible representations) of finite groups follows, and is sketched here in appendix A25; it says that all other invariant subspaces are obtained by weighted averages ('projections')

$$\rho^{(\mu)}(x) = \frac{d_\mu}{|G|}\sum_g \chi^{(\mu)}(g)\,U(g)\,\rho(x) = \frac{d_\mu}{|G|}\sum_g \chi^{(\mu)}(g)\,\rho(D(g^{-1})x) \qquad (25.10)$$

The above $\rho^{(A_1)}(x)$ invariant subspace is a special case, with all $\chi^{(A_1)}(g) = 1$.

By now the group acts in many different ways, so let us recapitulate:

g	abstract group element, multiplies other elements
$D(g)$	$[d\times d]$ state space transformation matrix, multiplies $x \in \mathcal{M}$
$U(g)$	operator, acts on functions $\rho(x)$ defined over state space $\mathcal{M}$
$D^{(\mu)}(g)$	$[d_\mu\times d_\mu]$ irrep, acts on invariant subspace $x \in \mathcal{M}^{(\mu)}$
$D^{reg}(g)$	$[\lvert G\rvert\times\lvert G\rvert]$ regular matrix rep, acts on vectors $x \in \mathcal{M}^{reg}$

Note that the state space transformation $D(g) \neq D(e)$ can leave sets of '*boundary*' points invariant (or 'invariant points', see (10.9)); for example, under reflection σ across a symmetry plane, the plane itself remains invariant. The boundary periodic orbits that belong to such pointwise invariant sets will require special care in evaluations of trace formulas.

25.3 Dynamics in the fundamental domain

What happens in the fundamental domain, stays in the fundamental domain.

—Professore Dottore Gatto Nero

How does a group act on the evolution operator $\mathcal{L}^t(y,x)$? As in (25.2), its value should be the same if evaluated at the same points in the rotated coordinates,

$$U(g)\mathcal{L}^t(y,x) = \mathcal{L}^t(D(g)^{-1}y, D(g)^{-1}x)\ . \qquad (25.11)$$

We are interested in a dynamical system invariant under the symmetry group G, i.e., with equations of motion invariant (equivariant) under all symmetries $g \in G$, section 10.1

$$D(g)\, f^t(x) = f^t(D(g)\, x)\,, \qquad (25.12)$$

hence for the evolution operator defined by (20.24) (we can omit the observable weight with no loss of generality, as long as the observable does not break the symmetry):

$$\begin{aligned} U(g^{-1})\, \mathcal{L}^t(y,x) &= \mathcal{L}^t(D(g)\, y, D(g)\, x) \\ &= \delta\Big(D(g)\, y - f^t(D(g)\, x)\Big) = \delta\Big(D(g)\,(y - f^t(x))\Big) \\ &= \frac{1}{|\det D(g)|}\, \delta\Big(y - f^t(x)\Big)\,. \end{aligned}$$

For compact groups $|\det D(g)| = 1$ by (10.3), so the evolution operator $\mathcal{L}^t(y,x)$ is *invariant* under group actions,

$$U(g)\, \mathcal{L}^t(y,x) = \mathcal{L}^t(y,x)\,. \qquad (25.13)$$

This is as it should be. If G is a symmetry of dynamics, the law that moves densities around should have the same form in all symmetry related coordinate systems.

As the function $\rho(x)$ that the evolution operator (20.24) acts on is now replaced by the regular basis vector of functions (25.4) over the fundamental domain, the evolution operator itself becomes a $[|G|{\times}|G|]$ *matrix*. If the initial point lies in tile $\hat{\mathcal{M}}_a$, its deterministic trajectory lands in the unique tile $\hat{\mathcal{M}}_b$, with a unique relative shift $g = b^{-1}a$, with the only non-vanishing entry $\mathcal{L}^t(y,x)_{ba} = \mathcal{L}^t(D(b)\hat{y}, D(a)\hat{x})$ wherever the regular representation $D^{reg}(g)_{ba}$ has entry 1 in row a and column b. Using the evolution operator invariance (25.13) one can move the end point y into the fundamental domain, and then use the relation $g = b^{-1}a$ to relate the start point x to its image in the fundamental domain,

$$\mathcal{L}^t(D(b)\hat{y}, D(a)\hat{x}) = \mathcal{L}^t(\hat{y}, D(g)\hat{x}) \equiv \hat{\mathcal{L}}^t(\hat{y}, \hat{x}; g)\,.$$

For a given g all non-vanishing entries are the same, and the evolution operator (20.24) is replaced by the $[|G|{\times}|G|]$ matrix of form

$$\mathcal{L}^t_{ba}(\hat{y}, \hat{x}; g) = D^{reg}(g)_{ba} \hat{\mathcal{L}}^t(\hat{y}, \hat{x}; g)\;,$$

if $\hat{x} \in \hat{\mathcal{M}}_a$ and $\hat{y} \in \hat{\mathcal{M}}_b$, zero otherwise, and the evolution $\hat{\mathcal{L}}^t(\hat{y}, \hat{x}; g)$ restricted to $\hat{\mathcal{M}}$. Another way to say it is that the law of evolution in the fundamental domain is given by

$$\hat{x}(t) = \hat{f}^t(\hat{x}_0) = D(g(t))\, f^t(\hat{x}_0)\,,$$

where the matrix $D(g(t))$ is the group operation that maps the end point of the full state space trajectory $x(t)$ back to its fundamental domain representative $\hat{x}(t)$. While the global trajectory runs over the full space $\mathcal{M}$, the symmetry-reduced trajectory is brought back into the fundamental domain $\hat{\mathcal{M}}$ every time it crosses into an adjoining tile; the two trajectories are related by the 'reconstruction' operation $g = g(\hat{x}_0, t)$ which maps the global trajectory endpoint into its fundamental domain image.

Now the traces (22.3) required for the evaluation of the eigenvalues of the evolution operator can be computed on the fundamental domain alone

$$\mathrm{tr}\,\mathcal{L}^t = \int_{\mathcal{M}} dx\,\mathcal{L}^t(x,x) = \sum_g^G \mathrm{tr}\,D^{reg}(g) \int_{\hat{\mathcal{M}}} d\hat{x}\hat{\mathcal{L}}^t(\hat{x},\hat{x};g)\,. \qquad (25.14)$$

Nothing seems to have been gained: the trace of regular representation matrix $\mathrm{tr}\,D^{reg}(g) = |G|\,\delta_{g,e}$ guarantees that only those repeats of the fundamental domain cycles $\hat{p}$ that correspond to complete global cycles p contribute, and the factor $\mathrm{tr}\,D^{reg}(e) = |G|$ simply says that integral over whole state space is $|G|$ times the integral over the fundamental domain.

example 25.10
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But not so fast! Nobody said that the traces of the *irreps*, $\mathrm{tr}\,D^{(\mu)}(g) = \chi^{(\mu)}(g)$, in the decomposition (25.9) are nonvanishing only for the identity operation e; they pick up a contribution for every reconstruction operation $g(\hat{x}_0, t)$,

$$\mathrm{tr}\,\mathcal{L}^t = \sum_\mu d_\mu\,\mathrm{tr}\,\hat{\mathcal{L}}^t_\mu\,, \quad \mathrm{tr}\,\hat{\mathcal{L}}^t_\mu = \sum_g^G \chi^{(\mu)}(g) \int_{\hat{\mathcal{M}}} d\hat{x}\,\hat{\mathcal{L}}^t(\hat{x},\hat{x};g)\,, \qquad (25.15)$$

and then the fundamental domain trace $\int d\hat{x}\,\hat{\mathcal{L}}^t(\hat{x},\hat{x};g)$ picks up a contribution from each fundamental domain prime cycle $\hat{p}$, i.e., all *relative periodic orbits*

$$\hat{x}_{\hat{p}} = g_{\hat{p}} f^{T_{\hat{p}}}(\hat{x}_{\hat{p}})\,, \qquad g_{\hat{p}} = g(\hat{x}_{\hat{p}}, T_{\hat{p}})\,.$$

In chapter 11 we have shown that a discrete symmetry induces degeneracies among periodic orbits and decomposes periodic orbits into repetitions of irreducible segments; this reduction to a fundamental domain furthermore leads to a convenient symbolic dynamics compatible with the symmetry, and, most importantly, to a factorization of dynamical zeta functions. This we now develop, first in a general setting and then for specific examples.

25.4 Discrete symmetry factorizations

As we saw in chapter 11, discrete symmetries relate classes of periodic orbits and reduce dynamics to a fundamental domain. Such symmetries simplify and improve the cycle expansions in a rather beautiful way; in classical dynamics, just as in quantum mechanics, the symmetrized subspaces can be probed by linear operators of different symmetries. If a linear operator commutes with the symmetry, it can be block-diagonalized, and, as we shall now show, the associated spectral determinants and dynamical zeta functions factorize.

We start by working out the factorization of dynamical zeta functions for reflection-symmetric systems in sect. 25.5, and the factorization of the corresponding spectral determinants in example 25.9. As reflection symmetry is essentially the only discrete symmetry that a map of the interval can have, this example completes the group-theoretic factorization of determinants and zeta functions for 1-dimensional maps.

25.4.1 Factorization of dynamical zeta functions

Let p be the full orbit, $\hat{p}$ the orbit in the fundamental domain and $h_{\hat{p}}$ an element of $\mathcal{H}_p$, the symmetry group of p. Restricting the volume integrations to the infinitesimal neighborhoods of the cycles p and $\hat{p}$, respectively, and performing the standard resummations, we obtain the identity

$$(1 - t_p)^{m_p} = \det\left(1 - D^{reg}(h_{\hat{p}})t_{\hat{p}}\right) , \tag{25.16}$$

valid cycle by cycle in the Euler products (22.11) for the dynamical zeta function. Here 'det' refers to the $[|G|\times|G|]$ regular matrix representation $D^{reg}(h_{\hat{p}})$; as we shall see, this determinant can be evaluated in terms of irrep characters, and no explicit representation of $D^{reg}(h_{\hat{p}})$ is needed. Finally, if a cycle p is invariant under the symmetry subgroup $\mathcal{H}_p \subseteq G$ of order h_p, its weight can be written as a repetition of a fundamental domain cycle

$$t_p = t_{\hat{p}}^{h_p} \tag{25.17}$$

computed on the irreducible segment that corresponds to a fundamental domain cycle.

According to (25.16) and (25.17), the contribution of a degenerate class of global cycles (cycle p with multiplicity $m_p = |G|/h_p$) to a dynamical zeta function is given by the corresponding fundamental domain cycle $\hat{p}$:

$$(1 - t_{\hat{p}}^{h_p})^{m_p} = \det\left(1 - D^{reg}(g_{\hat{p}})t_{\hat{p}}\right) \tag{25.18}$$

Let $D^{reg}(g) = \bigoplus_\mu d_\mu D^{(\mu)}(g)$ be the decomposition of the regular matrix representation into the d_μ-dimensional irreps μ of a finite group G. Such decompositions are block-diagonal, so the corresponding contribution to the Euler product (22.8) factorizes as

$$\det(1 - D^{reg}(g)t) = \prod_\mu \det(1 - D^{(\mu)}(g)t)^{d_\mu} , \tag{25.19}$$

where now the product extends over all distinct d_μ-dimensional irreps, each contributing d_μ times. For the cycle expansion purposes, it has been convenient to emphasize that the group-theoretic factorization can be effected cycle by cycle, as in (25.18); but from the evolution operator point of view, the key observation is that the symmetry reduces the evolution operator to a block diagonal form; this block diagonalization implies that the dynamical zeta functions (22.11) factorize as

$$\frac{1}{\zeta} = \prod_{\mu} \frac{1}{\zeta_{\mu}^{d_\mu}} , \qquad \frac{1}{\zeta_\mu} = \prod_{\hat{p}} \det\left(1 - D^{(\mu)}(g_{\hat{p}}) t_{\hat{p}}\right) . \tag{25.20}$$

Determinants of d-dimensional irreps can be evaluated using the expansion of determinants in terms of traces,

$$\begin{aligned} \det(1+M) &= 1 + \operatorname{tr} M + \frac{1}{2}\left((\operatorname{tr} M)^2 - \operatorname{tr} M^2\right) \\ &+ \frac{1}{6}\left((\operatorname{tr} M)^3 - 3\,(\operatorname{tr} M)(\operatorname{tr} M^2) + 2 \operatorname{tr} M^3\right) \\ &+ \cdots + \frac{1}{d!}\left((\operatorname{tr} M)^d - \cdots\right) , \end{aligned} \tag{25.21}$$

and each factor in (25.19) can be evaluated by looking up the characters $\chi^{(\mu)}(g) = \operatorname{tr} D^{(\mu)}(g)$ in standard tables [12]. In terms of characters, we have for the 1-dimensional representations

$$\det(1 - D^{(\mu)}(g)t) = 1 - \chi^{(\mu)}(g)t \ ,$$

for the 2-dimensional representations

$$\det(1 - D^{(\mu)}(g)t) = 1 - \chi^{(\mu)}(g)t + \frac{1}{2}\left(\chi^{(\mu)}(g)^2 - \chi^{(\mu)}(g^2)\right) t^2,$$

and so forth.

In the fully symmetric subspace $\operatorname{tr} D_{A_1}(g) = 1$ for all orbits; hence a straightforward fundamental domain computation (with no group theory weights) always yields a part of the full spectrum. In practice this is the most interesting subspectrum, as it contains the leading eigenvalue of the evolution operator. exercise 25.2

25.4.2 Factorization of spectral determinants

Factorization of the full spectral determinant (22.3) proceeds in essentially the same manner as the factorization of dynamical zeta functions outlined above. By (25.14) the trace of the evolution operator $\mathcal{L}^t$ splits into the sum of inequivalent irreducible subspace contributions $\sum_\mu \operatorname{tr} \mathcal{L}^t_\mu$, with

$$\operatorname{tr} \mathcal{L}^t_\mu = d_\mu \sum_{g \in G} \chi^{(\mu)}(g) \int_{\hat{\mathcal{M}}} d\hat{x}\, \mathcal{L}^t(D(g)^{-1}\hat{x}, \hat{x})\,.$$

This leads by standard manipulations to the factorization of (22.8) into

$$\begin{aligned} F(z) &= \prod_\mu F_\mu(z)^{d_\mu} \\ F_\mu(z) &= \exp\left(-\sum_{\hat{p}} \sum_{r=1}^{\infty} \frac{1}{r} \frac{\chi^{(\mu)}(g^r_{\hat{p}}) z^{n_{\hat{p}} r}}{|\det\left(\mathbf{1} - \hat{M}^r_{\hat{p}}\right)|}\right) , \end{aligned} \tag{25.22}$$

where $\hat{M}_{\hat{p}} = D(g_{\hat{p}})M_{\hat{p}}$ is the fundamental domain Jacobian. Boundary orbits require special treatment, discussed in sect. 25.4.3, with examples given in the next section as well as in the specific factorizations discussed below.

25.4.3 Boundary orbits

Before we can turn to a presentation of the factorizations of dynamical zeta functions for the different symmetries we have to discuss an effect that arises for orbits that run on a symmetry line that borders a fundamental domain. In our 3-disk example, no such orbits are possible, but they exist in other systems, such as in the bounded region of the Hénon-Heiles potential and in 1-d maps. For the symmetrical 4-disk billiard, there are in principle two kinds of such orbits, one kind bouncing back and forth between two diagonally opposed disks and the other kind moving along the other axis of reflection symmetry; the latter exists for bounded systems only. While there are typically very few boundary orbits, they tend to be among the shortest orbits, and their neglect can seriously degrade the convergence of cycle expansions, as those are dominated by the shortest cycles.

While such orbits are invariant under some symmetry operations, their neighborhoods are not. This affects the Jacobian matrix M_p of the linearization perpendicular to the orbit and thus the eigenvalues. Typically, *e.g.* if the symmetry is a reflection, some eigenvalues of M_p change sign. This means that instead of a weight $1/\det(\mathbf{1} - M_p)$ as for a regular orbit, boundary cycles also pick up contributions of form $1/\det(\mathbf{1} - D(g)M_p)$, where $D(g)$ is a symmetry operation that leaves the orbit pointwise invariant; see example 25.9.

Consequences for the dynamical zeta function factorizations are that sometimes a boundary orbit does not contribute. A derivation of a dynamical zeta function (22.11) from a determinant like (22.8) usually starts with an expansion of the determinants of the Jacobian. The leading order terms just contain the product of the expanding eigenvalues and lead to the dynamical zeta function (22.11). Next to leading order terms contain products of expanding and contracting eigenvalues and are sensitive to their signs. Clearly, the weights t_p in the dynamical zeta function will then be affected by reflections in the Poincaré section perpendicular to the orbit. In all our applications it was possible to implement these effects by the following simple prescription.

If an orbit is invariant under a little group $\mathcal{H}_p = \{e, b_2, \ldots, b_h\}$, then the corresponding group element in (25.16) will be replaced by a projector. If the weights are insensitive to the signs of the eigenvalues, then this projector is

$$g_p = \frac{1}{h}\sum_{i=1}^{h} b_i\,. \tag{25.23}$$

In the cases that we have considered, the change of sign may be taken into account by defining a sign function $\epsilon_p(g) = \pm 1$, with the “-" sign if the symmetry element

g flips the neighborhood. Then (25.23) is replaced by

$$g_p = \frac{1}{h}\sum_{i=1}^{h} \epsilon(b_i)\, b_i\,. \tag{25.24}$$

The factorizations (25.20), (25.22) are the central formulas of this chapter. We now work out the group theory factorizations of cycle expansions of dynamical zeta functions for C_2 and D_3 symmetries. D_2 and D_4 symmetries are worked out in appendix A25.

25.5 $C_2 = D_1$ factorization

As the simplest example of implementing the above scheme consider the $C_2 = D_1$ symmetry. For our purposes, all that we need to know here is that each orbit or configuration is uniquely labeled by an infinite string $\{s_i\}$, $s_i = +,-$ and that the dynamics is invariant under the $+ \leftrightarrow -$ interchange, i.e., it is C_2 symmetric. The C_2 symmetry cycles separate into two classes, the self-dual configurations $+-$, $++--$, $+++---$, $+--+-++-$, $\cdots$, with multiplicity $m_p = 1$, and the asymmetric configurations $+$, $-$, $++-$, $--+$, $\cdots$, with multiplicity $m_p = 2$. For example, as there is no absolute distinction between the "up" and the "down" spins, or the "left" or the "right" lobe, $t_+ = t_-$, $t_{++-} = t_{+--}$, and so on.

exercise 25.6

The symmetry reduced labeling $\rho_i \in \{0, 1\}$ is related to the standard $s_i \in \{+,-\}$ Ising spin labeling by

$$\begin{aligned} &\text{If} \quad s_i = s_{i-1} \quad \text{then} \quad \rho_i = 1 \\ &\text{If} \quad s_i \neq s_{i-1} \quad \text{then} \quad \rho_i = 0 \end{aligned} \tag{25.25}$$

For example, $\overline{+} = \cdots++++\cdots$ maps into $\cdots111\cdots = \overline{1}$ (and so does $\overline{-}$), $\overline{-+} = \cdots-+-+\cdots$ maps into $\cdots000\cdots = \overline{0}$, $\overline{-++-} = \cdots--++--++\cdots$ maps into $\cdots0101\cdots = \overline{01}$, and so forth. A list of such reductions is given in table 15.1.

Depending on the maximal symmetry group $\mathcal{H}_p$ that leaves an orbit p invariant (see sect. 25.3 as well as example 25.9), the contributions to the dynamical zeta function factor as

$$\begin{aligned} & & & A_1 \qquad A_2 \\ \mathcal{H}_p = \{e\}: \quad & (1-t_{\hat p})^2 &=& (1-t_{\hat p})(1-t_{\hat p}) \\ \mathcal{H}_p = \{e,\sigma\}: \quad & (1-t_{\hat p}^2) &=& (1-t_{\hat p})(1+t_{\hat p})\,, \end{aligned} \tag{25.26}$$

For example:

$$\begin{aligned} \mathcal{H}_{++-} = \{e\}: \quad & (1-t_{++-})^2 &=& (1-t_{001})(1-t_{001}) \\ \mathcal{H}_{+-} = \{e,\sigma\}: \quad & (1-t_{+-}) &=& (1-t_0)\ (1+t_0), \quad t_{+-} = t_0^2 \end{aligned}$$

This yields two binary cycle expansions. The A_1 subspace dynamical zeta function is given by the standard binary expansion (23.8). The antisymmetric A_2 subspace dynamical zeta function ζ_{A_2} differs from ζ_{A_1} only by a minus sign for cycles with an odd number of 0's:

$$\begin{aligned} 1/\zeta_{A_2} &= (1+t_0)(1-t_1)(1+t_{10})(1-t_{100})(1+t_{101})(1+t_{1000}) \\ &\quad (1-t_{1001})(1+t_{1011})(1-t_{10000})(1+t_{10001}) \\ &\quad (1+t_{10010})(1-t_{10011})(1-t_{10101})(1+t_{10111})\ldots \\ &= 1+t_0-t_1+(t_{10}-t_1t_0)-(t_{100}-t_{10}t_0)+(t_{101}-t_{10}t_1) \\ &\quad -(t_{1001}-t_1t_{001}-t_{101}t_0+t_{10}t_0t_1)-\ldots\ldots \end{aligned} \tag{25.27}$$

Note that the group theory factors do not destroy the curvature corrections (the cycles and pseudo cycles are still arranged into shadowing combinations).

If the system under consideration has a boundary orbit (*cf.* sect. 25.4.3) with group-theoretic factor $h_p = (e+\sigma)/2$, the boundary orbit does not contribute to the antisymmetric subspace

$$\begin{array}{lcll} & & A_1 & A_2 \\ \text{boundary:} \quad (1-t_p) & = & (1-t_{\hat{p}}) & (1-0t_{\hat{p}}) \end{array} \tag{25.28}$$

This is the $1/\zeta$ part of the boundary orbit factorization discussed in example 25.9, where the factorization of the corresponding spectral determinants for the 1-dimensional reflection symmetric maps is worked out in detail.

example 25.9
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25.6 D_3 factorization: 3-disk game of pinball

The next example, the D_3 symmetry, can be worked out by a glance at figure 15.12 (a). For the symmetric 3-disk game of pinball the fundamental domain is bounded by a disk segment and the two adjacent sections of the symmetry axes that act as mirrors (see figure 15.12 (b)). The three symmetry axes divide the space into six copies of the fundamental domain. Any trajectory on the full space can be pieced together from bounces in the fundamental domain, with symmetry axes replaced by flat mirror reflections. The binary $\{0,1\}$ reduction of the ternary three disk $\{1,2,3\}$ labels has a simple geometric interpretation: a collision of type 0 reflects the projectile to the disk it comes from (back–scatter), whereas after a collision of type 1 projectile continues to the third disk. For example, $\overline{23} = \cdots 232323 \cdots$ maps into $\cdots 000 \cdots = \overline{0}$ (and so do $\overline{12}$ and $\overline{13}$), $\overline{123} = \cdots 12312 \cdots$ maps into $\cdots 111 \cdots = \overline{1}$ (and so does $\overline{132}$), and so forth. A list of such reductions for short cycles is given in table 15.2.

D_3 has two 1-dimensional irreps, symmetric and antisymmetric under reflections, denoted A_1 and A_2, and a pair of degenerate 2-dimensional representations of mixed symmetry, denoted E. The contribution of an orbit with symmetry g to the $1/\zeta$ Euler product (25.19) factorizes according to

$$\det(1 - D^{reg}(h)t) = \left(1 - \chi^{(A_1)}(h)t\right)\left(1 - \chi^{(A_2)}(h)t\right)\left(1 - \chi^{(E)}(h)t + \chi^{(A_2)}(h)t^2\right)^2 \qquad (25.29)$$

with the three factors contributing to the D_3 irreps A_1, A_2 and E, respectively, and the 3-disk dynamical zeta function factorizes into $\zeta = \zeta_{A_1}\zeta_{A_2}\zeta_E^2$. Substituting the D_3 characters [12]

D_3	A_1	A_2	E
e	1	1	2
C, C^2	1	1	−1
σ_v	1	−1	0

into (25.29), we obtain for the three classes of possible orbit symmetries (indicated in the first column)

$$\begin{array}{rlcl} & g_{\hat{p}} & & \quad A_1 \qquad A_2 \qquad E \\ e: & (1-t_{\hat{p}})^6 & = & (1-t_{\hat{p}})(1-t_{\hat{p}})(1-2t_{\hat{p}}+t_{\hat{p}}^2)^2 \\ C,C^2: & (1-t_{\hat{p}}^3)^2 & = & (1-t_{\hat{p}})(1-t_{\hat{p}})(1+\ t_{\hat{p}}+t_{\hat{p}}^2)^2 \\ \sigma_v: & (1-t_{\hat{p}}^2)^3 & = & (1-t_{\hat{p}})(1+t_{\hat{p}})(1+0t_{\hat{p}}-t_{\hat{p}}^2)^2. \end{array} \qquad (25.30)$$

where σ_v stands for any one of the three reflections.

The Euler product (22.11) on each irreducible subspace follows from the factorization (25.30). On the symmetric A_1 subspace the ζ_{A_1} is given by the standard binary curvature expansion (23.8). The antisymmetric A_2 subspace ζ_{A_2} differs from ζ_{A_1} only by a minus sign for cycles with an odd number of 0's, and is given in (25.27). For the mixed-symmetry subspace E the curvature expansion is given by

$$\begin{aligned} 1/\zeta_E &= (1 + zt_1 + z^2t_1^2)(1 - z^2t_0^2)(1 + z^3t_{100} + z^6t_{100}^2)(1 - z^4t_{10}^2) \\ &\quad (1 + z^4t_{1001} + z^8t_{1001}^2)(1 + z^5t_{10000} + z^{10}t_{10000}^2) \\ &\quad (1 + z^5t_{10101} + z^{10}t_{10101}^2)(1 - z^5t_{10011})^2 \ldots \\ &= 1 + zt_1 + z^2(t_1^2 - t_0^2) + z^3(t_{001} - t_1t_0^2) \\ &\quad + z^4\left[t_{0011} + (t_{001} - t_1t_0^2)t_1 - t_{01}^2\right] \\ &\quad + z^5\left[t_{00001} + t_{01011} - 2t_{00111} + (t_{0011} - t_{01}^2)t_1 + (t_1^2 - t_0^2)t_{100}\right] + \cdots \end{aligned} \qquad (25.31)$$

We have reinserted the powers of z in order to group together cycles and pseudocycles of the same length. Note that the factorized cycle expansions retain the curvature form; long cycles are still shadowed by (somewhat less obvious) combinations of pseudo-cycles.

Referring back to the topological polynomial (18.40) obtained by setting $t_p = 1$, we see that its factorization is a consequence of the D_3 factorization of the ζ function:

$$1/\zeta_{A_1} = 1 - 2z\,, \quad 1/\zeta_{A_2} = 1\,, \quad 1/\zeta_E = 1 + z\,, \tag{25.32}$$

as obtained from (23.8), (25.27) and (25.31) for $t_p = 1$.

Their symmetry is $K = \{e, \sigma\}$, so according to (25.23), they pick up the group-theoretic factor $g_p = (e + \sigma)/2$. If there is no sign change in t_p, then evaluation of $\det(1 - \frac{\mathbf{e}+\sigma}{2} t_{\hat{p}})$ yields

$$\text{boundary:}\quad (1 - t_p)^3 = \overset{A_1}{(1 - t_{\hat{p}})}\overset{A_2}{(1 - 0t_{\hat{p}})}\overset{E}{(1 - t_{\hat{p}})^2}\,, \quad t_p = t_{\hat{p}}\,. \tag{25.33}$$

However, if the cycle weight changes sign under reflection, $t_{\sigma\hat{p}} = -t_{\hat{p}}$, the boundary orbit does not contribute to the subspace symmetric under reflection across the orbit;

$$\text{boundary:}\quad (1 - t_p)^3 = \overset{A_1}{(1 - 0t_{\hat{p}})}\overset{A_2}{(1 - t_{\hat{p}})}\overset{E}{(1 - t_{\hat{p}})^2}\,, \quad t_p = t_{\hat{p}}\,. \tag{25.34}$$

Résumé

If a dynamical system has a discrete symmetry, the symmetry should be exploited; much is gained, both in understanding of the spectra and ease of their evaluation. Once this is appreciated, it is hard to conceive of a calculation without factorization; it would correspond to quantum mechanical calculations without wave–function symmetrizations.

While the reformulation of the chaotic spectroscopy from the trace sums to the cycle expansions does not reduce the exponential growth in number of cycles with the cycle length, in practice only the short orbits are used, and for them the labor saving is dramatic. For example, for the 3-disk game of pinball there are 256 periodic points of length 8, but reduction to the fundamental domain non-degenerate prime cycles reduces the number of the distinct cycles of length 8 to 30.

In addition, cycle expansions of the symmetry reduced dynamical zeta functions converge dramatically faster than the unfactorized dynamical zeta functions. One reason is that the unfactorized dynamical zeta function has many closely spaced zeros and zeros of multiplicity higher than one; since the cycle expansion is a polynomial expansion in topological cycle length, accommodating such behavior requires many terms. The dynamical zeta functions on separate subspaces have more evenly and widely spaced zeros, are smoother, do not have symmetry-induced multiple zeros, and fewer cycle expansion terms (short cycle truncations) suffice to determine them. Furthermore, the cycles in the fundamental domain sample state space more densely than in the full space. For example, for the 3-disk problem, there are 9 distinct (symmetry unrelated) cycles of length 7 or less in full space, corresponding to 47 distinct periodic points. In the fundamental domain, we have 8 (distinct) periodic orbits up to length 4 and thus 22 different periodic points in 1/6-th the state space, i.e., an increase in density by a factor 3 with the same numerical effort.

We emphasize that the symmetry factorization (25.30) of the dynamical zeta function is *intrinsic* to the classical dynamics, and not a special property of quantal spectra. The factorization is not restricted to the Hamiltonian systems, or only to the configuration space symmetries; for example, the discrete symmetry can be a symmetry of the Hamiltonian phase space [16]. In conclusion, the manifold advantages of the symmetry reduced dynamics should thus be obvious; full state space cycle expansions, such as those of exercise 25.4, are useful only for cross-checking purposes.

Commentary

Remark 25.1. Symmetry reductions in periodic orbit theory. Some of the standard references on characters and irreps of compact groups are refs. [2, 6, 8, 12, 21]. We found Tinkham [19] introduction to the basic concepts the most enjoyable.

This chapter is based on a collaborative effort with B. Eckhardt. The group-theoretic factorizations of dynamical zeta functions that we develop here were first introduced and applied in ref. [3]. They are closely related to the symmetrizations introduced by Gutzwiller [10] in the context of the semiclassical periodic orbit trace formulas, put into more general group-theoretic context by Robbins [16], whose exposition, together with Lauritzen's [13] treatment of the boundary orbits, has influenced the presentation given here. The symmetry reduced trace formula for a finite symmetry group $G = \{e, g_2, \ldots, g_{|G|}\}$ with $|G|$ group elements, where the integral over Haar measure is replaced by a finite group discrete sum $|G|^{-1}\sum_{g\in G} = 1$, derived in ref. [3]. A related group-theoretic decomposition in context of hyperbolic billiards was utilized in ref. [1], and for the Selberg's zeta function in ref. [20]. One of its loftier antecedents is the Artin factorization formula of algebraic number theory, which expresses the zeta-function of a finite extension of a given field as a product of L-functions over all irreps of the corresponding Galois group.

The techniques of this chapter have been applied to computations of the 3-disk classical and quantum spectra in refs. [7, 17], and to a "Zeeman effect" pinball and the x^2y^2 potentials in ref. [4, 5]. In a larger perspective, the factorizations developed above are special cases of a general approach to exploiting the group-theoretic invariances in spectra computations, such as those used in enumeration of periodic geodesics [15, 18] for hyperbolic billiards [9] and Selberg zeta functions [11].

Remark 25.2. Other symmetries. In addition to the symmetries exploited here, time reversal symmetry and a variety of other non-trivial discrete symmetries can induce further relations among orbits; we shall point out several of examples of cycle degeneracies under time reversal. We do not know whether such symmetries can be exploited for further improvements of cycle expansions.

References

[1] N. L. Balasz and A. Voros, "Chaos on the pseudosphere", Phys. Rep. **143**, 109–240 (1986).

[2] J. F. Cornwell, *Group Theory in Physics: An Introduction* (Academic, New York, 1997).

[3] P. Cvitanović and B. Eckhardt, "Symmetry decomposition of chaotic dynamics", Nonlinearity **6**, 277–311 (1993).

[4] P. Dahlqvist, "Determination of resonance spectra for bound chaotic systems", J. Phys. A **27**, 763–785 (1994).

[5] P. Dahlqvist and G. Russberg, "Periodic orbit quantization of bound chaotic systems", J. Phys. A **24**, 4763–4778 (1991).

[6] M. S. Dresselhaus, G. Dresselhaus, and A. Jorio, *Group Theory: Application to the Physics of Condensed Matter* (Springer, New York, 2007).

[7] B. Eckhardt, G. Russberg, P. Cvitanović, P. E. Rosenqvist, and P. Scherer, "Pinball scattering", in *Quantum chaos: between order and disorder*, edited by G. Casati and B. Chirikov (Cambridge Univ. Press, Cambridge, 1995), p. 483.

[8] J. P. Elliott and P. G. Dawber, *Symmetry in Physics*, Vol. 2 (MacMillan, Surrey, 1979).

[9] M. C. Gutzwiller, "Phase-integral approximation in momentum space and the bound states of an atom", J. Math. Phys. **8**, 1979–2000 (1967).

[10] M. C. Gutzwiller, "The quantization of a classically ergodic system", Physica D **5**, 183–207 (1982).

[11] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, New York, 1990).

[12] M. Hamermesh, *Group Theory and Its Application to Physical Problems* (Dover, New York, 1962).

[13] B. Lauritzen, "Discrete symmetries and the periodic-orbit expansions", Phys. Rev. A **43**, 603–606 (1991).

[14] E. N. Lorenz, "Deterministic nonperiodic flow", J. Atmos. Sci. **20**, 130–141 (1963).

[15] H. P. McKean, "Selberg's trace formula as applied to a compact Riemann surface", Commun. Pure Appl. Math. **25**, 225–246 (1972).

[16] J. M. Robbins, "Discrete symmetries in periodic-orbit theory", Phys. Rev. A **40**, 2128–2136 (1989).

[17] P. Scherer, Quantenzustände eines Klassisch Chaotischen Billards, PhD thesis (Univ. Köln, 1991).

[18] M. Sieber and F. Steiner, "Classical and quantum mechanics of a strongly chaotic billiard system", Physica D **44**, 248–266 (1990).

[19] M. Tinkham, *Group Theory and Quantum Mechanics* (Dover, New York, 2003).

[20] A. Venkov and P. Zograf, "On analogues of the Artin factorization formulas in the spectral theory of automorphic functions connected with induced representations of Fuchsian groups", Math. USSR **21**, 435–443 (1983).

[21] E. P. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Academic, New York, 1931).

25.7 Examples

Example 25.1. A matrix representation of 2-element group C_2: If a 2-dimensional map $f(x)$ has the symmetry $x_1 \to -x_1$, $x_2 \to -x_2$, the symmetry group G consists of the identity and $C = C^{1/2}$, a rotation by π around the origin. The map f must then commute with rotations by π, $f(D(C)x) = D(C)f(x)$, with the matrix representation of C given by the $[2 \times 2]$ matrix

$$D(C) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} . \qquad (25.35)$$

C satisfies $C^2 = e$ and can be used to decompose the state space into mutually orthogonal symmetric and antisymmetric subspaces by means of projection operators (25.50).
(continued in example 25.3)

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Example 25.2. A matrix representation of cyclic group C_3: A 3-dimensional matrix representation of the 3-element cyclic group $C_3 = \{e, C^{1/3}, C^{2/3}\}$ is given by the three rotations by $2\pi/3$ around z-axis in a 3-dimensional state space,

$$D(e) = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad D(C^{1/3}) = \begin{bmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} & \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} & \\ & & 1 \end{bmatrix},$$

$$D(C^{2/3}) = \begin{bmatrix} \cos\frac{4\pi}{3} & -\sin\frac{4\pi}{3} & \\ \sin\frac{4\pi}{3} & \cos\frac{4\pi}{3} & \\ & & 1 \end{bmatrix}.$$

(continued in example 25.4) (X. Ding)

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Example 25.3. A 2-tiles state space: The state space $\mathcal{M} = \{x_1\text{-}x_2 \text{ plane}\}$ of example 25.1, with symmetry group $G = \{e, C\}$, can be tiled by a fundamental domain $\hat{\mathcal{M}} = \{\text{half-plane } x_1 \geq 0\}$, and $C\hat{\mathcal{M}} = \{\text{half-plane } x_1 \leq 0\}$, its image under rotation by π.

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Example 25.4. The regular representation of cyclic group C_3: (continued from example 25.2) Take an arbitrary function $\rho(x)$ over the state space $x \in \mathcal{M}$, and define a fundamental domain $\hat{\mathcal{M}}$ as a 1/3 wedge, with axis z as its (symmetry invariant) edge. The state space is tiled with three copies of the wedge,

$$\mathcal{M} = \hat{\mathcal{M}}_1 \cup \hat{\mathcal{M}}_2 \cup \hat{\mathcal{M}}_3 = \hat{\mathcal{M}} \cup C^{1/3}\hat{\mathcal{M}} \cup C^{2/3}\hat{\mathcal{M}} .$$

Function $\rho(x)$ can be written as the 3-dimensional vector of functions over the fundamental domain $\hat{x} \in \hat{\mathcal{M}}$,

$$(\rho_1^{reg}(\hat{x}), \rho_2^{reg}(\hat{x}), \rho_3^{reg}(\hat{x})) = (\rho(\hat{x}), \rho(C^{1/3}\hat{x}), \rho(C^{2/3}\hat{x})) . \qquad (25.36)$$

The multiplication table of C_3 is given in table 25.2. By (25.5), the regular representation matrices $D^{reg}(g)$ have '1' at the location of g in the multiplication table, '0' elsewhere. The actions of the operator $U(g)$ are now represented by permutations matrices (blank entries are zeros):

$$D^{reg}(e) = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad D^{reg}(C^{1/3}) = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}, \quad D^{reg}(C^{2/3}) = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}. \qquad (25.37)$$

(X. Ding)

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C_2	e	σ
e	e	σ
σ^{-1}	σ	e

C_3	e	$C^{1/3}$	$C^{2/3}$
e	e	$C^{1/3}$	$C^{2/3}$
$(C^{1/3})^{-1}$	$C^{2/3}$	e	$C^{1/3}$
$(C^{2/3})^{-1}$	$C^{1/3}$	$C^{2/3}$	e

Table 25.2: The multiplication tables of the 2-element group C_2, and C_3, the group of symmetries of a 3-blade propeller.

C_2	e	σ
A	1	1
B	1	-1

C_3	e	$C^{1/3}$	$C^{2/3}$
0	1	1	1
1	1	ω	ω^2
2	1	ω^2	ω

D_3	e	3σ	$2C$
A	1	1	1
B	1	-1	1
E	2	0	-1

Table 25.3: C_2, C_3 and D_3 character tables. The classes $\{\sigma_{12}, \sigma_{13}, \sigma_{14}\}$, $\{C^{1/3}, C^{2/3}\}$ are denoted 3σ, $2C$, respectively.

Example 25.5. The regular representation of dihedral group D_3: The multiplication table of D_3 is given in table 25.1. By (25.5), the 6 regular representation matrices $D^{reg}(g)$ have '1' at the location of g in the multiplication table, '0' elsewhere. For example, the regular representation of the action of operators $U(\sigma_{23})$ and $U(C^{2/3})$ are given in table 25.1. (X. Ding)

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Example 25.6. Irreps of cyclic group C_3: (continued from example 25.4) We would like to generalize the symmetric-antisymmetric functions decomposition of C_2 to the order 3 group C_3. Symmetrization can be carried out on any number of functions, but there is no obvious 'antisymmetrization'. We draw instead inspiration from the Fourier transformation for a finite periodic lattice, and construct from the regular basis (25.36) a new set of basis functions

$$\rho_0(\hat{x}) = \frac{1}{3}\left[\rho(\hat{x}) + \rho(C^{1/3}\hat{x}) + \rho(C^{2/3}\hat{x})\right] \tag{25.38}$$

$$\rho_1(\hat{x}) = \frac{1}{3}\left[\rho(\hat{x}) + \omega\,\rho(C^{1/3}\hat{x}) + \omega^2\rho(C^{2/3}\hat{x})\right] \tag{25.39}$$

$$\rho_2(\hat{x}) = \frac{1}{3}\left[\rho(\hat{x}) + \omega^2\rho(C^{1/3}\hat{x}) + \omega\,\rho(C^{2/3}\hat{x})\right]. \tag{25.40}$$

The representation of group C_3 in this new basis is block diagonal by inspection:

$$D(e) = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad D(C^{1/3}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \quad D(C^{2/3}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{bmatrix}. \tag{25.41}$$

Here $\omega = e^{2i\pi/3}$. So C_3 has three 1-dimensional irreps ρ_0, ρ_1 and ρ_2. Generalization to any C_n is immediate: this is just a finite lattice Fourier transform. (X. Ding)

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Example 25.7. Character table of D_3: (continued from example 25.5) Let us construct table 25.3. Spectroscopists conventions are to use labels A and B for symmetric, respectively antisymmetric nondegenerate irreps, and E for the doubly degenerate irreps. So 1-dimensional representations are denoted by A and B, depending on whether the basis function is symmetric or antisymmetric with respect to transpositions σ_{ij}. E denotes the

2-dimensional representation. As D_3 has 3 classes, the dimension sum rule $d_1^2+d_2^2+d_3^2 = 6$ has only one solution $d_1 = d_2 = 1$, $d_3 = 2$. Hence there are two 1-dimensional irreps and one 2-dimensional irrep. The first column is $1, 1, 2$, and the first row is $1, 1, 1$ corresponding to the 1-d symmetric representation. We take two approaches to figure out the remaining 4 entries. First, since B is antisymmetric 1-d representation, so the characters should be ± 1. We anticipate $\chi^{(B)}(\sigma) = -1$ and can quickly figure out the remaining 3 positions. We check the obtained table satisfies the orthonormal relations. Second, denote $\chi^{(B)}(\sigma) = x$ and $\chi^{(E)}(\sigma) = y$, then from the orthonormal relation of the second column with the first column and itself, we obtain $1 + x + 2y = 0$, and $1 + x^2 + y^2 = 6/3$, we get two sets of solutions, one of them can be shown not compatible with other orthonormality relations, so $x = -1$, $y = 0$. Similarly, we can get the other two characters. (X. Ding) click to return: p. 470

Example 25.8. Basis for irreps of D_3: (continued from example 25.7) From table 25.3, we have

$$P^A\rho(x) = \frac{1}{6}\left[\rho(x) + \rho(\sigma_{12}x) + \rho(\sigma_{23}x) + \rho(\sigma_{31}x) + \rho(C^{1/3}x) + \rho(C^{2/3}x)\right] \tag{25.42}$$

$$P^B\rho(x) = \frac{1}{6}\left[\rho(x) - \rho(\sigma_{12}x) - \rho(\sigma_{23}x) - \rho(\sigma_{31}x) + \rho(C^{1/3}x) + \rho(C^{2/3}x)\right] \tag{25.43}$$

For projection into irrep E, we need to figure out the explicit matrix representation first. Obviously, the following 2 by 2 matrices are E irrep.

$$D^E(e) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad D^E(C^{1/3}) = \begin{bmatrix}\omega & 0\\ 0 & \omega^2\end{bmatrix}, \quad D^E(C^{2/3}) = \begin{bmatrix}\omega^2 & 0\\ 0 & \omega\end{bmatrix} \tag{25.44}$$

$$D^E(\sigma_{12}) = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad D^E(\sigma_{23}) = \begin{bmatrix}0 & \omega^2\\ \omega & 0\end{bmatrix}, \quad D^E(\sigma_{31}) = \begin{bmatrix}0 & \omega\\ \omega^2 & 0\end{bmatrix} \tag{25.45}$$

So apply projection operator on $\rho(x)$ and $\rho(\sigma_{12}x)$:

$$P_1^E\rho(x) = \frac{1}{3}\left[\rho(x) + \omega\rho(C^{1/3}x) + \omega^2\rho(C^{2/3}x)\right] \tag{25.46}$$

$$P_2^E\rho(x) = \frac{1}{3}\left[\rho(x) + \omega^2\rho(C^{1/3}x) + \omega\rho(C^{2/3}x)\right] \tag{25.47}$$

$$P_1^E\rho(\sigma_{12}x) = \frac{1}{3}\left[\rho(\sigma_{12}x) + \omega\rho(\sigma_{31}x) + \omega^2\rho(\sigma_{23}x)\right] \tag{25.48}$$

$$P_2^E\rho(\sigma_{12}x) = \frac{1}{3}\left[\rho(\sigma_{12}x) + \omega^2\rho(\sigma_{31}x) + \omega\rho(\sigma_{23}x)\right] \tag{25.49}$$

Under the invariant basis

$$\{P^A\rho(x), P^B\rho(x), P_1^E\rho(x), P_2^E\rho(\sigma_{12}x), P_1^E\rho(\sigma_{12}x), P_2^E\rho(x)\}$$

,

$$D(\sigma_{23}) = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \omega^2 & 0 & 0\\ 0 & 0 & \omega & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \omega^2\\ 0 & 0 & 0 & 0 & \omega & 0\end{bmatrix} \quad D(C^{1/3}) = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & \omega & 0 & 0 & 0\\ 0 & 0 & 0 & \omega^2 & 0 & 0\\ 0 & 0 & 0 & 0 & \omega & 0\\ 0 & 0 & 0 & 0 & 0 & \omega^2\end{bmatrix}.$$

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Example 25.9. Reflection symmetric 1-d maps: Consider f, a map on the interval with reflection symmetry $f(-x) = -f(x)$. A simple example is the piecewise-linear sawtooth map of figure 11.1. Denote the reflection operation by $\sigma x = -x$. The symmetry of the map implies that if $\{x_n\}$ is a trajectory, than also $\{\sigma x_n\}$ is a trajectory because $\sigma x_{n+1} = \sigma f(x_n) = f(\sigma x_n)$. The dynamics can be restricted to a fundamental domain, in this case to one half of the original interval; every time a trajectory leaves this interval, it can be mapped back using σ. Furthermore, the evolution operator is invariant under the group, $U(\sigma)\mathcal{L}^t(y,x) = \mathcal{L}^t(y,x)$. σ satisfies $\sigma^2 = \mathbf{e}$ and can be used to decompose the state space into mutually orthogonal symmetric and antisymmetric subspaces by means of projection operators

$$
\begin{aligned}
P_{A_1} &= \frac{1}{2}(1+U(\sigma))\,, \qquad P_{A_2} = \frac{1}{2}(1-U(\sigma))\,, \\
\mathcal{L}^t_{A_1}(y,x) &= P_{A_1}\mathcal{L}^t(y,x) = \frac{1}{2}\left(\mathcal{L}^t(y,x)+\mathcal{L}^t(-y,x)\right)\,, \\
\mathcal{L}^t_{A_2}(y,x) &= P_{A_2}\mathcal{L}^t(y,x) = \frac{1}{2}\left(\mathcal{L}^t(y,x)-\mathcal{L}^t(-y,x)\right)\,. \qquad (25.50)
\end{aligned}
$$

To compute the traces of the symmetrization and antisymmetrization projection operators (25.50), we have to distinguish three kinds of cycles: asymmetric cycles a, symmetric cycles s built by repeats of irreducible segments $\tilde{s}$, and boundary cycles b. Now we show that the spectral determinant can be written as the product over the three kinds of cycles: $\det(1-\mathcal{L}^t) = \det(1-\mathcal{L}^t)_a \det(1-\mathcal{L}^t)_{\tilde{s}} \det(1-\mathcal{L}^t)_b$.

Asymmetric cycles: A periodic orbits is not symmetric if $\{x_a\}\cap\{\sigma x_a\} = \emptyset$, where $\{x_a\}$ is the set of periodic points belonging to the cycle a. Thus σ generates a second orbit with the same number of points and the same stability properties. Both orbits give the same contribution to the first term and no contribution to the second term in (25.50); as they are degenerate, the prefactor $1/2$ cancels. Resuming as in the derivation of (22.11) we find that asymmetric orbits yield the same contribution to the symmetric and the antisymmetric subspaces:

$$
\det(1-\mathcal{L}_\pm)_a = \prod_a \prod_{k=0}^{\infty}\left(1-\frac{t_a}{\Lambda_a^k}\right), \qquad t_a = \frac{z^{n_a}}{|\Lambda_a|}\,.
$$

Symmetric cycles: A cycle s is reflection symmetric if operating with σ on the set of periodic points reproduces the set. The period of a symmetric cycle is always even ($n_s = 2n_{\tilde{s}}$) and the mirror image of the x_s periodic point is reached by traversing the irreducible segment $\tilde{s}$ of length $n_{\tilde{s}}$, $f^{n_{\tilde{s}}}(x_s) = \sigma x_s$. $\delta(x - f^n(x))$ picks up $2n_{\tilde{s}}$ contributions for every even traversal, $n = rn_{\tilde{s}}$, r even, and $\delta(x + f^n(x))$ for every odd traversal, $n = rn_{\tilde{s}}$, r odd. Absorb the group-theoretic prefactor in the Floquet multiplier by defining the stability computed for a segment of length $n_{\tilde{s}}$,

$$
\Lambda_{\tilde{s}} = -\left.\frac{\partial f^{n_{\tilde{s}}}(x)}{\partial x}\right|_{x=x_s}\,.
$$

Restricting the integration to the infinitesimal neighborhood $\mathcal{M}_s$ of the s cycle, we obtain the contribution to $\mathrm{tr}\,\mathcal{L}^n_\pm$:

$$\begin{aligned} z^n \mathrm{tr}\,\mathcal{L}^n_\pm &\to \int_{\mathcal{M}_s} dx\, z^n\, \frac{1}{2}\, (\delta(x - f^n(x)) \pm \delta(x + f^n(x))) \\ &= n_{\tilde{s}} \left(\sum_{r=2}^{\text{even}} \delta_{n,rn_{\tilde{s}}} \frac{t^r_{\tilde{s}}}{1 - 1/\Lambda^r_{\tilde{s}}} \pm \sum_{r=1}^{\text{odd}} \delta_{n,rn_{\tilde{s}}} \frac{t^r_{\tilde{s}}}{1 - 1/\Lambda^r_{\tilde{s}}} \right) \\ &= n_{\tilde{s}} \sum_{r=1}^{\infty} \delta_{n,rn_{\tilde{s}}} \frac{(\pm t_{\tilde{s}})^r}{1 - 1/\Lambda^r_{\tilde{s}}} \,. \end{aligned}$$

Substituting all symmetric cycles s into $\det(1 - \mathcal{L}_\pm)$ and resuming we obtain:

$$\det(1 - \mathcal{L}_\pm)_{\tilde{s}} = \prod_{\tilde{s}} \prod_{k=0}^{\infty} \left(1 \mp \frac{t_{\tilde{s}}}{\Lambda^k_{\tilde{s}}}\right)$$

Boundary cycles: In the example at hand there is only one cycle which is neither symmetric nor antisymmetric, but lies on the boundary of the fundamental domain, the fixed point at the origin. Such cycle contributes simultaneously to both $\delta(x - f^n(x))$ and $\delta(x + f^n(x))$:

$$\begin{aligned} z^n \mathrm{tr}\,\mathcal{L}^n_\pm &\to \int_{\mathcal{M}_b} dx\, z^n\, \frac{1}{2}\, (\delta(x - f^n(x)) \pm \delta(x + f^n(x))) \\ &= \sum_{r=1}^{\infty} \delta_{n,r}\, t^r_b\, \frac{1}{2} \left(\frac{1}{1 - 1/\Lambda^r_b} \pm \frac{1}{1 + 1/\Lambda^r_b} \right) \\ z^n\, \mathrm{tr}\,\mathcal{L}^n_+ &\to \sum_{r=1}^{\infty} \delta_{n,r} \frac{t^r_b}{1 - 1/\Lambda^{2r}_b}\,; \qquad z^n\, \mathrm{tr}\,\mathcal{L}^n_- \to \sum_{r=1}^{\infty} \delta_{n,r} \frac{1}{\Lambda^r_b} \frac{t^r_b}{1 - 1/\Lambda^{2r}_b}\,. \end{aligned}$$

Boundary orbit contributions to the factorized spectral determinants follow by resummation:

$$\det(1 - \mathcal{L}_+)_b = \prod_{k=0}^{\infty} \left(1 - \frac{t_b}{\Lambda^{2k}_b}\right), \qquad \det(1 - \mathcal{L}_-)_b = \prod_{k=0}^{\infty} \left(1 - \frac{t_b}{\Lambda^{2k+1}_b}\right)$$

Only the even derivatives contribute to the symmetric subspace, and only the odd ones to the antisymmetric subspace, because the orbit lies on the boundary.

Finally, the symmetry reduced spectral determinants follow by collecting the above results:

$$F_+(z) = \prod_a \prod_{k=0}^{\infty} \left(1 - \frac{t_a}{\Lambda^k_a}\right) \prod_{\tilde{s}} \prod_{k=0}^{\infty} \left(1 - \frac{t_{\tilde{s}}}{\Lambda^k_{\tilde{s}}}\right) \prod_{k=0}^{\infty} \left(1 - \frac{t_b}{\Lambda^{2k}_b}\right)$$

$$F_-(z) = \prod_a \prod_{k=0}^{\infty} \left(1 - \frac{t_a}{\Lambda^k_a}\right) \prod_{\tilde{s}} \prod_{k=0}^{\infty} \left(1 + \frac{t_{\tilde{s}}}{\Lambda^k_{\tilde{s}}}\right) \prod_{k=0}^{\infty} \left(1 - \frac{t_b}{\Lambda^{2k+1}_b}\right) \qquad (25.51)$$

exercise 25.1
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Example 25.10. 3-disk billiard / D_3 cycle weights factorized: Compare, for example, the contributions of the $\overline{12}$ and $\overline{0}$ cycles of figure 15.12. $\mathrm{tr}\, D^{reg}(h)\hat{\mathcal{L}}$ does not get a contribution from the $\overline{0}$ cycle, as the symmetry operation that maps the first half of the $\overline{12}$ into the fundamental domain is a reflection, and $\mathrm{tr}\, D^{reg}(\sigma) = 0$. In contrast, $\sigma^2 = e$, $\mathrm{tr}\, D^{reg}(\sigma^2) = 6$ insures that the repeat of the fundamental domain fixed point $\mathrm{tr}\,(D^{reg}(h)\hat{\mathcal{L}})^2 = 6t_0^2$, gives the correct contribution to the global trace $\mathrm{tr}\,\mathcal{L}^2 = 3 \cdot 2t_{12}$.

We see by inspection in figure 15.12 that $t_{12} = t_0^2$ and $t_{123} = t_1^3$.

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Exercises

25.1. **Sawtooth map desymmetrization.** Work out the some of the shortest global cycles of different symmetries and fundamental domain cycles for the sawtooth map of figure 11.1. Compute the dynamical zeta function and the spectral determinant of the Perron-Frobenius operator for this map; check explicitly the factorization (25.51).

25.2. **2-dimensional asymmetric representation.** The above expressions can sometimes be simplified further using standard group-theoretical methods. For example, the $\frac{1}{2}\left((\mathrm{tr}\, M)^2 - \mathrm{tr}\, M^2\right)$ term in (25.21) is the trace of the antisymmetric part of the $M \times M$ Kronecker product. Show that if α is a 2-dimensional representation, this is the A_2 antisymmetric representation, and

$$\text{2-dim:} \quad \det(1 - D_\alpha(h)t) = 1 - \chi_\alpha(h)t + \chi_{A_2}(h)t^2. \quad (25.52)$$

25.3. **Characters of D_3.** (continued from exercise 10.4) $D_3 \cong C_{3v}$, the group of symmetries of an equilateral triangle: has three irreducible representations, two one-dimensional and the other one of multiplicity 2.

(a) All finite discrete groups are isomorphic to a permutation group or one of its subgroups, and elements of the permutation group can be expressed as cycles. Express the elements of the group D_3 as cycles. For example, one of the rotations is (123), meaning that vertex 1 maps to 2, $2 \to 3$, and $3 \to 1$.

(b) Use your representation from exercise 10.4 to compute the D_3 character table.

(c) Use a more elegant method from the group-theory literature to verify your D_3 character table.

(d) Two D_3 irreducible representations are one dimensional and the third one of multiplicity 2 is formed by $[2\times2]$ matrices. Find the matrices for all six group elements in this representation.

(Hint: get yourself a good textbook, like Hamermesh [12] or Tinkham [19], and read up on classes and characters.)

25.4. **3–disk unfactorized zeta cycle expansions.** Check that the curvature expansion (23.3) for the 3-disk pinball, assuming no symmetries between disks, is given by

$$\begin{aligned} 1/\zeta &= (1 - z^2 t_{12})(1 - z^2 t_{13})(1 - z^2 t_{23}) \\ &\quad (1 - z^3 t_{123})(1 - z^3 t_{132})(1 - z^4 t_{1213}) \\ &\quad (1 - z^4 t_{1232})(1 - z^4 t_{1323})(1 - z^5 t_{12123}) \cdots \\ &= 1 - z^2 t_{12} - z^2 t_{23} - z^2 t_{31} - z^3 (t_{123} + t_{132}) \\ &\quad - z^4 [(t_{1213} - t_{12} t_{13}) + (t_{1232} - t_{12} t_{23}) \\ &\quad + (t_{1323} - t_{13} t_{23})] \qquad (25.53) \\ &\quad - z^5 [(t_{12123} - t_{12} t_{123}) + \cdots] - \cdots \end{aligned}$$

Show that the symmetrically arranged 3-disk pinball cycle expansion of the Euler product (23.3) (see table 18.5 and figure 11.2) is given by:

$$\begin{aligned} 1/\zeta &= (1 - z^2 t_{12})^3 (1 - z^3 t_{123})^2 (1 - z^4 t_{1213})^3 \\ &\quad (1 - z^5 t_{12123})^6 (1 - z^6 t_{121213})^6 \\ &\quad (1 - z^6 t_{121323})^3 \ldots \qquad (25.54) \\ &= 1 - 3z^2\, t_{12} - 2z^3\, t_{123} - 3z^4\, (t_{1213} - t_{12}^2) \\ &\quad - 6z^5\, (t_{12123} - t_{12} t_{123}) \\ &\quad - z^6\, (6\, t_{121213} + 3\, t_{121323} + t_{12}^3 - 9\, t_{12} t_{1213} - t_{123}^2) \\ &\quad - 6z^7\, (t_{1212123} + t_{1212313} + t_{1213123} + t_{12}^2 t_{123} \\ &\quad - 3\, t_{12} t_{12123} - t_{123} t_{1213}) \\ &\quad - 3z^8\, (2\, t_{12121213} + t_{12121313} + 2\, t_{12121323} \\ &\quad + 2\, t_{12123123} + 2\, t_{12123213} + t_{12132123} \\ &\qquad + 3\, t_{12}^2 t_{1213} + t_{12} t_{123}^2 - 6\, t_{12} t_{121213} \\ &\qquad - 3\, t_{12} t_{121323} - 4\, t_{123} t_{12123} - t_{1213}^2) - \cdots \end{aligned}$$

25.5. **3-disk desymmetrization.**

a) Work out the 3-disk symmetry factorization for the 0 and 1 cycles, i.e. which symmetry do they have, what is the degeneracy in full space and how do they factorize (how do they look in the A_1, A_2 and the E representations).

b) Find the shortest cycle with no symmetries and factorize it as in a)

c) Find the shortest cycle that has the property that its time reversal is not described by the same symbolic dynamics.

d) Compute the dynamical zeta functions and the spectral determinants (symbolically) in the three representations; check the factorizations (25.20) and (25.22).

(Per Rosenqvist)

25.6. **C_2 factorizations: the Lorenz and Ising systems.** In the Lorenz system [14] the labels + and − stand for the left or the right lobe of the attractor and the symmetry is a rotation by π around the z-axis. Similarly, the Ising Hamiltonian (in the absence of an external magnetic field) is invariant under spin flip. Work out the factorizations for some of the short cycles in either system.

25.7. **Ising model.** The Ising model with two states $\epsilon_i = \{+, -\}$ per site, periodic boundary condition, and Hamiltonian

$$H(\epsilon) = -J \sum_i \delta_{\epsilon_i, \epsilon_{i+1}} ,$$

is invariant under spin-flip: $+ \leftrightarrow -$. Take advantage of that symmetry and factorize the dynamical zeta function for the model, i.e., find all the periodic orbits that contribute to each factor and their weights.

25.8. **One orbit contribution.** If p is an orbit in the fundamental domain with symmetry h, show that it contributes to the spectral determinant with a factor

$$\det \left(1 - D^{reg}(h) \frac{t_p}{\lambda_p^k} \right) ,$$

where $D^{reg}(h)$ is the regular representation of G.